Anomalous Self-Similarity in a Turbulent Rapidly Rotating Fluid

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Our velocity measurements on quasi-two-dimensional turbulent flow in a rapidly rotating annulus yield self-similar (scale-independent) probability distribution functions for longitudinal velocity differences, $\delta v(\ell) = v(x + \ell) - v(x)$. These distribution functions are strongly non-Gaussian, suggesting that the coherent vortices play a significant role. The structure functions $\langle[\delta v(\ell)]^p\rangle \sim \ell^{\zeta_p}$ exhibit anomalous scaling: $\zeta_p = \frac{p}{2}$ rather than the expected $\zeta_p = \frac{p}{3}$. Correspondingly, the energy spectrum is described by $E(k) \sim k^{-2}$ rather than the expected $E(k) \sim k^{-5/3}$.

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In large scale flows in the earth's atmosphere and oceans or in gaseous planets, the Coriolis force dominates other forces and to lowest order is balanced by pressure gradients (geostrophic balance). The dimensionless number that characterizes this regime is the Rossby number, the ratio of magnitudes of the inertial term in the Navier-Stokes equation ($\vec{u} \cdot \nabla\vec{u}$, where $\vec{u}$ is the velocity) to the Coriolis term ($2\vec{\Omega} \times \vec{u}$, where Ω is the rotation rate). For planetary flows on large scales the Rossby number is typically in the range 0.05–0.2, and turbulence in such flows is quite different from three-dimensional (3D) turbulence in an inertial reference frame. Our experiments on a rotating annulus are the first to determine the statistical properties of turbulence in a low Rossby number flow.

One of the most significant effects of rapid rotation on a fluid is the two-dimensionalization of the flow [1]. Experiments have been recently conducted on quasi-2D turbulence in several nonrotating fluid systems [2], and simulations have been conducted on strictly 2D flows [3], but turbulence in a low Rossby number flow is different— it is a special mathematical limit on the 3D Navier-Stokes equations. Rapidly rotating flows contain large coherent vortices and jets, and the interplay between coherent vortical and/or wave motions and disorder is central to the understanding of quasi-2D turbulence in these flows.

A major question regarding turbulence is whether the statistics are self-similar across a wide range of spatial scales. This is usually investigated by examining the probability distribution function (PDF) of longitudinal velocity differences between two points of separation ℓ, $\delta v(\ell) = v(x + \ell) - v(x)$, where v is along the line connecting the two points. Self-similarity means that the PDFs have a functional form independent of the separation ℓ. Self-similarity can also be determined from the scaling of the velocity structure function, $S_p(\ell) \equiv \langle|\delta v(\ell)|^p\rangle \sim \ell^{\zeta_p}$: for any self-similar flow, ζ_p will vary linearly with p [1]. In particular, the theory for homogeneous isotropic turbulence predicts that if a flow is self-similar, there will exist an inertial range with $\zeta_p = p/3$ [1]. However, 3D turbulence is generally *not* self-similar [1]. In contrast, our experiments on low Rossby number turbulence reveal a self-similar flow with strongly non-Gaussian statistics and anomalous scaling of the structure function, $\zeta_p = p/2$ instead of $\zeta_p = p/3$. We now describe these experiments and results.

Our apparatus consists of an annular tank filled with water and covered with a solid lid; the inner diameter of the tank is 21.6 cm and the outer diameter is 86.4 cm [4]. The depth of the tank increases from 17.1 cm at the inner radius to 20.3 cm at the outer radius [5]. Flow in the annulus is produced by continuously pumping water in a closed circuit through two concentric rings of 120 holes each in the bottom of the tank; the source ring is at a radius of 18.9 cm and the sink ring is at 35.1 cm. The radially outward flux from the pumping couples with the Coriolis force to generate a counterrotating jet. The rapid rotation leads to a flow that is essentially 2D except in the thin Ekman boundary layers at the top and bottom surfaces [4].

In the present experiments the tank rotates at 11.0 rad/s, sufficiently fast to produce an essentially 2D flow [4], and the flux is 150 cm^3/s, sufficiently large to produce turbulent flow. The azimuthal velocity is measured using two hot film probes that are inserted through the top lid and extend 1 cm into the water on opposite sides of the tank, midway between the inner and outer walls. Each probe was sampled at 150 Hz for periods of two hours, giving 10^6 data points per probe for each run, and the measurements were repeated four times, yielding a total of 8×10^6 data points. Using the maximum velocity ($U_{\max} \simeq 22$ cm/s) as the velocity scale and the distance between the forcing rings ($L = 16.2$ cm) as the integral length scale, we calculate the Rossby number ($U_{\max}/2\Omega L$) to be 0.06 and the Reynolds number Re to be 35 000 (the Reynolds number based on the Taylor microscale is $\mathrm{Re}_\lambda = 360$).

Instantaneous 2D velocity fields were obtained using a particle image velocimetry (PIV) system with a horizontal light sheet at midfluid depth and a rotating camera above the tank. Data were collected in sets of 50 instantaneous velocity fields, equivalent to approximately 3×10^4 velocity values at the radius of the hot film probes. Although this sample size is inadequate for high order statistics, the spatial information provided by the PIV measurements complements the long velocity time series obtained with the hot film probes. We also made PIV measurements at the same rotation rate at a higher pumping rate of 550 cm^3/s (Re = 100 000).

Two vorticity profiles obtained with the PIV system are shown in Fig. 1. When the pump is first turned on, the flow consists of rings of small cyclonic and anticyclonic vortices that form above the sink and source rings, respectively. Small vortices of like sign merge and grow, and a counterrotating jet forms between the two rings. At long times the merging process leads to large vortices, which are larger at higher pumping rates (compare the two flows in Fig. 1).

We compute energy power spectra from the velocity time series data assuming Taylor's frozen turbulence hypothesis, which is applicable because the turbulent intensity (ratio of the rms velocity fluctuation to the mean velocity) is less than 10%. The spectra contain a region with $E(k) \sim k^{-2}$ (Fig. 2), in contrast with the Kolmogorov-Kraichnan prediction of $E(k) \sim k^{-5/3}$ [1] for the inverse (energy) cascade of 2D turbulence. Spectra obtained from PIV measurements of the azimuthal velocity at Re = 100 000 yield the same scaling as the time series data, although the PIV data are noisier (see upper curve in Fig. 2).

The probability distribution functions for velocity differences, $\delta v(\ell)$, reveal that the flow is self-similar: measurements for different separations ℓ collapse onto a single curve when normalized (Fig. 3). The PDF is far from Gaussian. The enhanced probability in the tails is likely due to the strong velocity differences [6] that arise as coherent vortices pass the probes. The standard deviation of the velocity differences is given by $\delta v_{\mathrm{rms}}(\ell) \sim \ell^{1/2}$, as seen in the insets of Fig. 3; this is consistent with the anomalous scaling found for the energy spectrum, $E(k) \sim k^{-2}$.

Despite the strong effects of rotation, our quasi-2D flow is nearly isotropic in the plane: the standard deviations of the azimuthal and radial components of the velocity at the radial position of the hot film probes (obtained from PIV measurements) are comparable, at 2.4 and 3.0 cm/s, respectively. The velocity increments $[\delta v_r(\ell)]_{\mathrm{rms}}$ and $[\delta v_\theta(\ell)]_{\mathrm{rms}}$ are also comparable for a wide range of separations ℓ in the azimuthal direction (e.g., Ref. [7]). The approximately circular shape of the coherent vortices in Fig. 1 is further evidence for local isotropy of the flow.

Now we examine the structure function scaling, plotting S_p as a function of ℓ, as shown in Fig. 4(a). Consider the region labeled A, from about 2 to 8 cm. While the range in scales is narrow, the data yield values for the scaling exponents ζ_p that are clearly different from the expected values: the slopes in region A of Fig. 4(a) are $p/2$ rather than $p/3$ as shown in curve A of Fig. 5. There is also a hint of another scaling region, labeled B in Fig. 4(a), but the

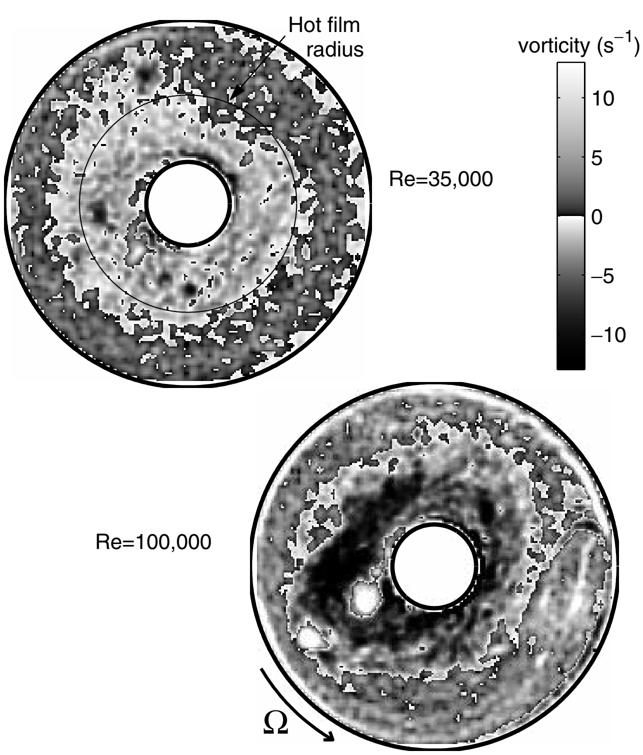

FIG. 1. Vorticity field at Reynolds number 35 000 ($Q = 150\ \text{cm}^3/\text{s}$) and 100 000 ($Q = 550\ \text{cm}^3/\text{s}$) for rotation rate 11.0 rad/s. Vortices with a light (dark) center are cyclonic (anticyclonic). The vortices are advected by the clockwise jet.

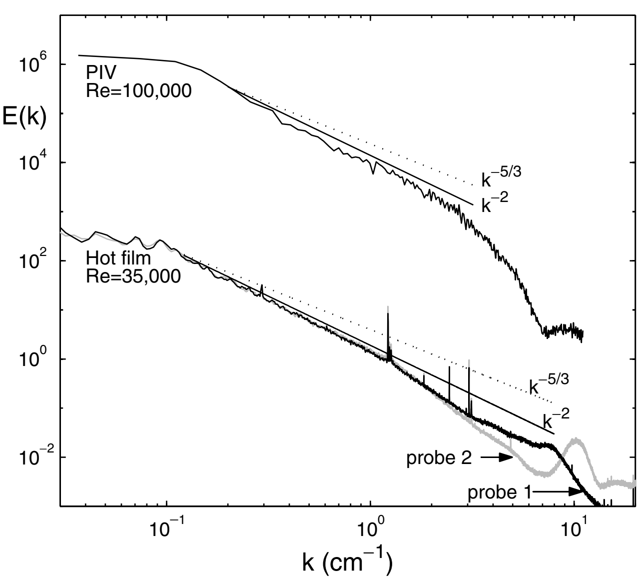

FIG. 2. Energy spectra (arb. units) with dotted lines showing the $k^{-5/3}$ inverse cascade and solid lines showing k^{-2} behavior. Fits of all of the hot film spectra in the range $0.1 < k < 1.22\ \text{cm}^{-1}$ give a slope of -2.04 ± 0.06. The sharp spectral peaks correspond to harmonics of the tank's rotation rate, not to dynamics of the flow.

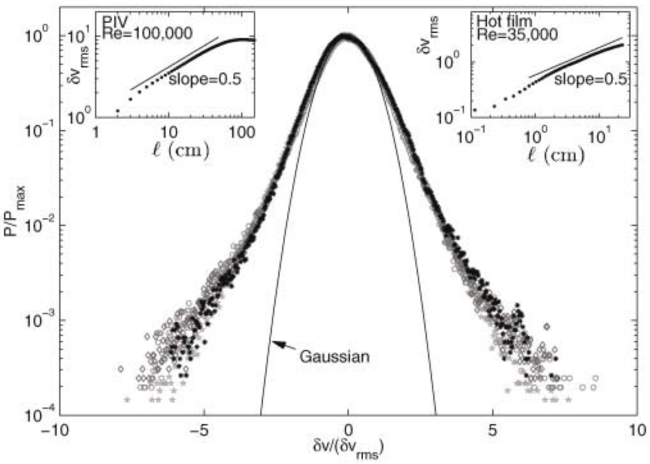

FIG. 3. Normalized probability distribution function for the velocity differences, demonstrating self-similar behavior: data for different separations ($\ell = 0.6, 4.6, 9.2, 17.3$ cm) collapse onto a single curve. The velocity differences are normalized by their standard deviation $(\delta v)_{\rm rms}$ and the probability by its maximum value, $P_{\max}$. The standard deviation of the velocity differences (see insets) scales as ℓ^{α}, where $\alpha = 0.50 \pm 0.06$ for Re $= 35\,000$ and $\alpha = 0.54 \pm 0.06$ for Re $= 100\,000$.

spatial range is too small to deduce a scaling exponent. The transition between the two regions can be seen more clearly in the plot of $S_{10}/\ell^{5.5}$ vs ℓ in the inset of Fig. 4(a) [8].

We also determine the *relative* structure function scaling using Benzi *et al.*'s extended self-similarity (ESS) method, where values of the exponent ratio ζ_p/ζ_3 are deduced from the slopes of log-log plots of S_p vs S_3 [9]. This approach extends the scaling range from 0.5 to 15 cm, as Fig. 4(b) illustrates, and the ratios ζ_p/ζ_3 have the value $p/3$, as they must for ESS scaling of a self-similar flow. The exponent values ζ_p deduced from the ℓ dependence of S_p [region A in Fig. 4(a)] and from the ESS plot [Fig. 4(b)] are compared in Fig. 5.

The only other experimental observation of self-similar quasi-2D flow was by Paret and Tabeling [10]. They observed near-Gaussian statistics and a Kolmogorov-Kraichnan $k^{-5/3}$ energy spectrum in an experiment on a nonrotating thin layer of electrolyte in a magnetic field [11]. A prediction of a steeper wave number dependence, $E(k) \sim k^{-2}$, has been made for a turbulent flow made up of Lundgren spiral vortices that were not allowed to stretch in the third dimension [12]. The k^{-2} energy spectrum has also been predicted by phenomenological theories for rotating turbulent flow [13]. Further, a recent direct numerical simulation by Smith and Waleffe [14] of the 3D Navier-Stokes equations showed that rapid rotation led to a quasi-2D flow with large scale vortices and an energy spectrum steeper than $k^{-5/3}$. We conclude therefore that rotation and/or strong vortices (see Fig. 1) are likely responsible for the anomalous k^{-2} energy spectrum that we observe in our experiments.

The k^{-2} energy spectrum apparently corresponds to the inverse energy cascade of 2D turbulence. In the early

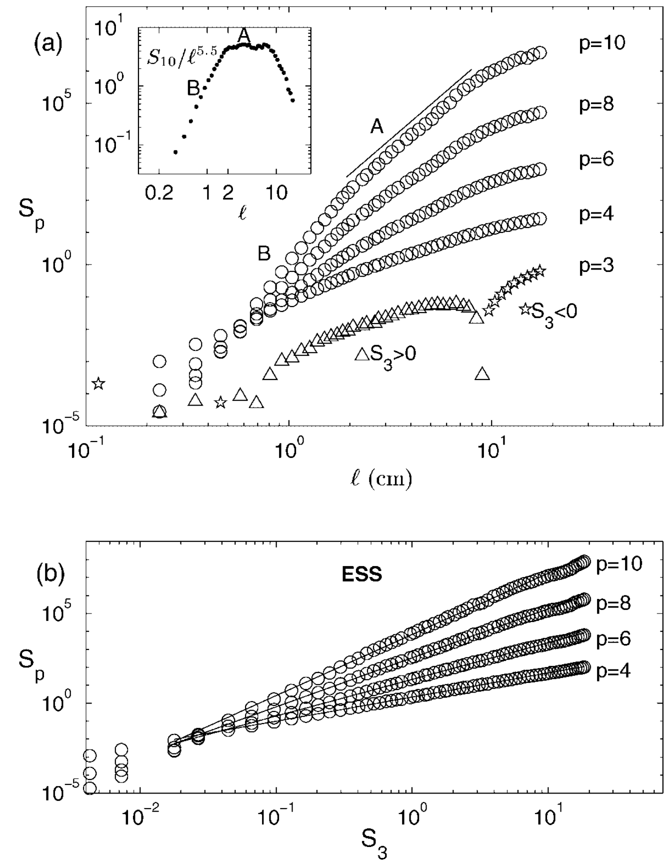

FIG. 4. Even order structure functions (a) as a function of ℓ and (b) as a function of S_3 (an extended self-similarity plot), for the hot film data at Re $= 35\,000$. The graph of $S_{10}/\ell^{5.5}$ in the inset in (a) emphasizes the sharpness of the bend in S_{10} at $\ell \simeq 2$ cm. The data in (b) were fit over the range $5 \times 10^{-2} < S_3 < 10$.

stages of formation of our flow, the inverse cascade is visible in the merging of small vortices to form larger vortices. This process becomes less apparent in the later stationary phase due to the presence of many other vortical structures. But evidence for the cascade direction is suggested by the sign of $S_3(\ell)$ since $S_3 > 0$ implies an inverse cascade in a homogeneous isotropic flow. In our flow, which is only weakly anisotropic in the plane, $S_3(\ell) > 0$ for $1 < \ell < 10$ cm. The lower bound of the range is the injection scale, corresponding to the distance between the source holes (1 cm); the upper bound varies with the pumping rate and is limited by the size of the system.

The $p/2$ value obtained for ζ_p in our flow is possibly a consequence of a cascade driven by a radial velocity shear. This process would yield an energy flux $\varepsilon \sim v_0(\delta v_{\|})^2/\ell$, where v_0 would be a velocity governing the time scale for the merging of vortices, $v_{\|}$ the azimuthal velocity, and ℓ the azimuthal separation. For a scale invariant ε, the resulting pth order velocity structure function would have the scaling exponent $p/2$.

In summary, this is the first laboratory study of the statistical properties of a rotating quasi-2D turbulent flow,

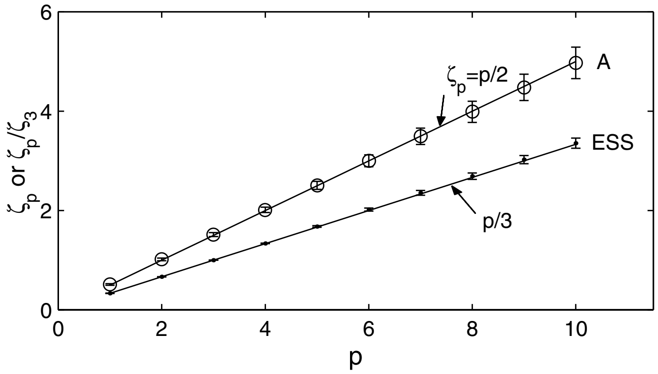

FIG. 5. Scaling exponents ζ_p as a function of p for region A of Fig. 4(a), and from the extended self-similarity analysis, Fig. 4(b). Region A, the inverse energy cascade region, yields $\zeta_p = (0.50 \pm 0.03)p$, while the ESS analysis yields $\zeta_p/\zeta_3 = (0.33 \pm 0.01)p$, in accordance with the results of the direct scaling in region A. The error bars show the standard deviation of the eight separate data sets of 10^6 points.

and the results show surprising differences from nonrotating quasi-2D flows: our velocity difference PDFs are non-Gaussian due to the presence of coherent vortices, yet the PDFs are self-similar, collapsing onto a single curve in the range $0.5 < \ell < 15$ cm. Previous reports of self-similar quasi-2D flow statistics showed very slight departures from Gaussianity and no large coherent structures. The self-similarity is also evident in the scaling of the structure function exponents, i.e., the exponent values increase linearly with order. However, our exponents are given by $\zeta_p = p/2$ instead of the expected $p/3$. Our results suggest that low Rossby number flows, such as large scale planetary flows, may exhibit statistics different from those of nonrotating 2D turbulence.

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[1] U. Frisch *Turbulence, the Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, UK, 1995).

[2] J. Paret and P. Tabeling, Phys. Rev. Lett. **79**, 4162 (1997); M. A. Rutgers, Phys. Rev. Lett. **81**, 2244 (1998); M. Rivera, P. Vorobieff, and R. Ecke, Phys. Rev. Lett. **81**, 1417 (1998).

[3] L. Smith and V. Yakhot, Phys. Rev. Lett. **71**, 352 (1993); H. Clercx and G. Van Heijst, Phys. Rev. Lett. **85**, 306 (2000); G. Boffetta, A. Celani, and M. Vergassola, Phys. Rev. E **61**, R29 (2000).

[4] T. H. Solomon, J. Holloway, and H. L. Swinney, Phys. Fluids **5**, 1971 (1993).

[5] The sloping bottom, a "beta plane," approximates to first order the variation of the Coriolis force with latitude.

[6] Z.-S. She, E. Jackson, and S. Orszag, Proc. R. Soc. London A **434**, 101 (1991).

[7] B. Dhruva, Y. Tsuji, and K. Sreenivasan, Phys. Rev. E **56**, R4928 (1997).

[8] The 2 cm lower limit of region A is below the 5 cm ($k = 1.2\ \mathrm{cm}^{-1}$ in Fig. 2) length, where the spectral behavior changes; this difference might be due to systematic error in the calibration of the probes, or perhaps because the scaling regions in real space and in Fourier space do not exactly correspond, as discussed in Ref. [1], (p. 62).

[9] R. Benzi, S. Ciliberto, R. Tripiccione, C. Baudet, F. Massaioli, and S. Succi, Phys. Rev. E **48**, R29 (1993).

[10] J. Paret and P. Tabeling, Phys. Fluids **10**, 3126 (1998).

[11] The only other experimental studies of scaling in quasi-2D turbulence were conducted on soap films, where departure from linear scaling of ζ_p vs p was observed, indicating that these flows were not self-similar. See P. Vorobieff, M. Rivera, and R. Ecke, Phys. Fluids **11**, 2167 (1999); W. B. Daniel and M. Rutgers (to be published).

[12] A. D. Gilbert, J. Fluid Mech. **193**, 475 (1988).

[13] Y. Zhou, Phys. Fluids **7**, 2092 (1995); V. M. Canuto and M. S. Dubovikov, Phys. Rev. Lett. **78**, 666 (1997).

[14] L. Smith and F. Waleffe, Phys. Fluids **11**, 1608 (1999).