

Hysteretic wave drag in shallow water

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During motion from deep to shallow water, multiple equilibria may emerge, each with identical drag. In this study, we show experimentally that the same drag force may be experienced at several different speeds of body motion in shallow water, and that this can be explained by a localised amplification of the wave drag near the shallow wave speed. Whilst wave drag in shallow water has been predicted with varying degrees of accuracy by several inviscid models, the bifurcation properties have not yet been discussed. Here, we address the nonlinear dynamics by considering the quasi-steady motion of a body between deep and shallow water, where the depth is slowly varying. In particular, we illustrate possible hysteresis routes between coexisting equilibrium states that have significantly different velocities, presenting interesting implications for naval and racing applications.

1. Introduction

The effect of bathymetry, or water depth, on the drag of an object moving through a body of water has numerous important applications, ranging from naval and coastal engineering to boat sports and swimming (Boucher *et al.* 2018; Fourdrinoy *et al.* 2019). It is well known that there is significant difference between the waves generated by a body moving in deep water and shallow water, and this is reflected in the drag properties. For example, as first discussed by Havelock (1922), wave drag may be either exacerbated or diminished in shallow water, depending on the speed regime. However, the catastrophic emergence of multiple equilibrium states in shallow water has not yet been discussed. In this study, we illustrate the existence of such states, both experimentally and theoretically, and use the example of quasi-steady motion in a body of water with slowly varying depth to explore the resulting nonlinear dynamics.

To distinguish between deep and shallow water regimes, a commonly used parameter is the non-dimensional ratio between the water depth h and the typical wave length λ . The dispersive properties of deep water ($h/\lambda \gg 1$) mean that the wave drag behaviour is largely dominated by a resonance between the boat speed u and the wave speed c (Havelock 1922). The boat speed is usually characterised by the length-based Froude number $Fr_L = u/\sqrt{gL}$, where L is the boat length. Wave drag is typically largest within a vicinity of $Fr_L \approx 0.5$, and its importance decays as $Fr_L \rightarrow 0, \infty$ (Michell 1898; Videler 2012). By contrast, the non-dispersive properties of shallow water ($h/\lambda \ll 1$) mean that no such resonance exists. Instead, since all waves travel at the same speed \sqrt{gh} , the only resonance is isolated to the vicinity where the depth-based Froude number is near unity $Fr_h = u/\sqrt{gh} \sim 1$. The collapse of this resonance has been compared to shock behaviour in hydraulic jumps, and supersonic transition in aeronautics (Tuck 1989).

The consequence is that when a boat moves from deep water to shallow water, and vice versa, there is an interesting and unintuitive change of behaviour, as first discussed by Havelock (1922) and later by various other authors (Tuck 1989; Hofman & Kozarski

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2000). There are three definitive regimes of interest: subcritical, critical and super-critical. The sub-critical region $Fr_h \ll 1$ corresponds to situations where the depth is sufficiently large that it makes little impact on the drag. The critical regime $Fr_h \sim 1$ exhibits resonance with the shallow wave speed, causing a drastic accentuation of the wave drag. By contrast, in the very shallow super-critical regime $Fr_h \gg 1$, the wave drag is less than it would be in infinitely deep water. Hence, as a body is pushed at constant force between deep and shallow water, it may undergo either an acceleration or a deceleration, depending on its speed.

There have been numerous studies of the effects of the bathymetry on the wave drag. One of the most notable is the linear theory of Havelock (1922), in which the wave resistance was calculated for a moving pressure disturbance at the surface of a body of water with finite depth. Another linear approach, derived by Sretensky (1937) and later summarised in English by Weinblum (1950), adjusted the infinite depth theory of Michell (1898) to account for both finite depth and width. Other studies have included nonlinear effects (Doctors & Dagan 1980), the effects of unsteadiness (Doctors 1975), shear currents (Li & Ellingsen 2016) and capillary waves (Wedołowski & Napiórkowski 2013). However, as mentioned above, what has not yet been discussed in the above literature is the existence of multiple equilibrium states and the nonlinear dynamics during motion between deep and shallow water. In particular, the associated bifurcation pattern has important consequences on boat racing applications, where different solution branches correspond to different speeds, and hence different race times.

In this study we consider the nonlinear dynamics of a boat pushed at constant force at the surface of a body of water with varying depth. We restrict our attention to the case where the length scale associated with depth changes is much larger than the length scale associated with acceleration/deceleration, similarly to Gourlay (2003). Such an approximation is realistic in a large number of practical cases. In this way, it is acceptable to consider the wave field steady at each instantaneous depth, and we focus on the quasi-steady dynamics of the boat (where the only important time-dependence is that associated with depth change). Using a tow-tank experiment, we measure the steady drag at various different speeds and depths, and we illustrate that multiple solutions emerge in the shallow water regime. This sort of bifurcation is reminiscent of the stick-slip problem studied by Baumberger & Caroli (2006).

We compare our tow-tank results to the experimental data from the European Development Centre for Inland and Coastal Navigation, Duisburg (also known as the VBD reports), as detailed by Hofman (2006), as well as surveying the existing theoretical descriptions of the wave drag. These include the linear theories of Sretensky (1937), Havelock (1922), and our recent boundary layer description of Benham *et al.* (2019), adjusted to account for finite depth. We employ the validated theory to explore the bifurcation patterns of the system, using the two relevant bifurcation parameters: the forcing and the depth. Then, we describe the quasi-steady motion of a boat in a lake with a sinusoidal depth profile as a case study, revealing interesting speed dynamics and possible hysteresis routes.

2. Motion in deep and shallow water

We consider the motion of a body of mass m moving in a large body of water with finite depth. Written in terms of the Cartesian coordinate system (x, y, z) , the body motion is purely in the x -direction. The z -axis is defined such that the resting water surface is given by $z = 0$, and the water subsurface only varies in the direction of motion $z = -h(x)$. The body is driven forwards with a constant force F , and the position and speed $x(t), u(t)$

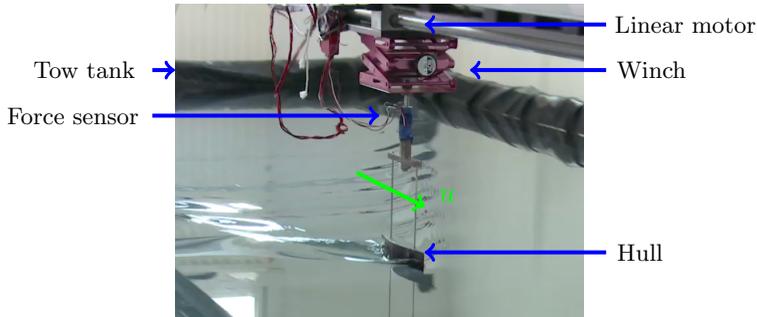


Figure 1: Labelled photograph of our experimental setup: a hull is pulled through the water at constant velocity by a linear motor and connected via supporting bars to a force sensor. The water depth for each experiment is uniform.

are governed by the dynamics

$$\dot{x} = u, \quad (2.1)$$

$$(m + m_a)\dot{u} = F - R_d(x, u), \quad (2.2)$$

where R_d is the drag force, which depends on both space (via the water depth) and the boat speed, and m_a is the added mass (Newman 2018). We make the key assumption that the depth is sufficiently slowly varying that any acceleration/deceleration due to depth change is small, or equivalently $\dot{u}/g \ll 1$. In this way, the drag term in (2.2) is approximated as its steady value at each instantaneous depth and speed (i.e. ignoring any dependence on acceleration).

As is often done, we decompose the drag into three different components: the wave drag, form drag and skin drag, such that

$$R_d = R_w + R_f + R_s. \quad (2.3)$$

The wave drag R_w is the force associated with the sustained generation of waves due to body motion, and depends greatly on both the speed and the draft (Benham *et al.* 2019), as well as the depth of the water (Havelock 1922). Typical speed values for different types of boats and aquatic animals are given by Boucher *et al.* (2018), and depth ratios can often be as small as $h/L = 0.2$. The form drag R_f is the force due to the surrounding pressure field and is effectively a measure of how streamlined an object is. Finally, the the skin drag R_s relates to the viscous friction between the wetted body surface and the surrounding fluid. As we will discuss later, the form and skin drag have a relatively weak dependence on depth and speed compared to the wave drag. Therefore, we make the simplifying assumption that R_f and R_s are constants for the range of depths and speeds we consider, whereas R_w depends on both $u(t)$ and $h(x(t))$.

We measure the total drag force R_d on a 3D-printed hull experimentally by pulling it through a large basin of water at constant velocity using a linear motor (see Figure 1). We use a symmetric hull, taken from the family of shapes proposed by Benham *et al.* (2019). The shape varies only in the direction of motion x , not in the vertical z , and is hence represented by a function $y = f(x)$, where the exact closed form expression can be found in (Benham *et al.* 2019). The dimensions of the submerged[†] hull in the (x, y, z) directions are $(L, W, H) = (18, 3, 2.5)$ cm, giving aspect ratios in the horizontal and vertical directions as $\alpha = L/W = 6$ and $\beta = L/H = 7.2$, respectively.

[†] Note that the total hull height is 5 cm, but only half of it is submerged beneath the water.

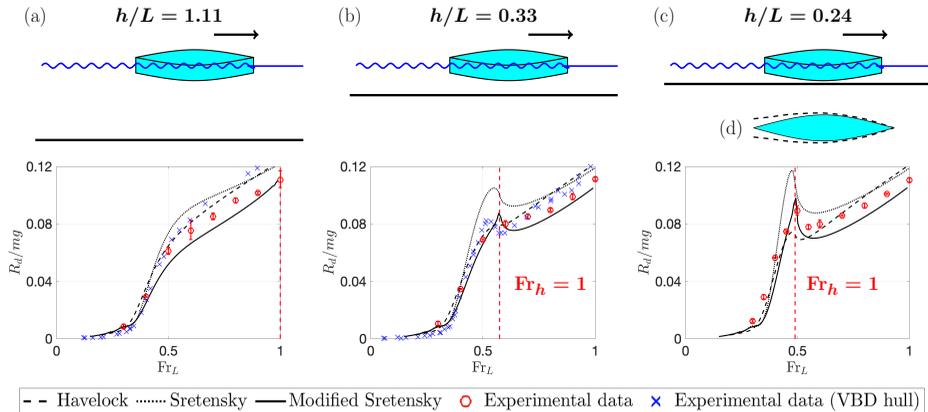


Figure 2: (a,b,c) Comparison between experimental measurements and theoretical predictions of the drag force for different depths and Froude numbers, and an illustration of each depth case above. The critical depth-based Froude number $Fr_h = u/\sqrt{gh} = 1$ is indicated in each plot. An illustration of the calculated boundary layer profile for our hull shape, which we use in our modification to Sretensky’s theory, is given in (d).

The hull is connected to the motor by two supporting bars and a force/displacement sensor. The experiments are repeated at least 3 times for accuracy. We do not allow the pitch of the boat to change, and we do not study the effects of planing (Rabaud & Moisy 2014). We study several different values of the water depth, $h/L = \{1.11, 0.81, 0.5, 0.33, 0.24\}$, and Fr_L values between 0.2 and 1, and in each case the water depth is uniform throughout the basin. These measurements for uniform depth will be used later to approximate the instantaneous drag in a medium with slowly varying depth (under the quasi-steady assumption).

In Figure 2 we plot the drag force R_d for various different values of the water depth h/L and the length-based Froude number Fr_L (though we omit the cases $h/L = 0.5, 0.81$ since they have effectively no depth effects, and are very similar to $h/L = 1.11$). In each case, we plot the drag in terms of the typical weight needed to balance the Archimedes buoyancy force of a boat of such volume $mg = \rho g L^3 / \alpha \beta$. Error bars correspond to one standard deviation of the time signal given by the force sensor. In addition to our drag measurements, we also plot the drag measurements taken from the VBD reports, as detailed by Hofman (2006), for which a Taylor Standard series ship hull shape is used (given in terms of a fifth order polynomial (Gertler 1954)). For the latter case, we only display the data in the cases where the same depth ratio was available. There are also several theoretical results overlaid on the same plot, which will be discussed shortly.

For depths larger than $h/L \approx 0.4$, the drag behaves similarly to the case of infinite depth, with a monotone increasing curve. However, for depths smaller than this, resonance with the shallow water wave speed accentuates the drag in an isolated region near $Fr_h = 1$, causing a pinching of the curve. This results in the emergence of a stationary point (maximum) near $Fr_h = 1$, and another stationary point (minimum) for a slightly larger Fr_L . Indeed, there are multiple values of the length-based Froude number that result in the same drag force. This implies that by pushing a boat with constant force in the critical regime, as many as three different velocities could be achieved.

The next step is to compare these results to available theoretical predictions. A full derivation of the theories used here can be found in the original references, though for our purposes we simply state the formulae. Two commonly used formulae for the wave drag

are the linear theories of Sretensky (1937) and Havelock (1922). Sretensky's formula, which is an extension of Michell's formula for the wave drag at infinite depth (Michell 1898), assumes an inviscid, irrotational, steady flow past a slender body. The wave drag is given in non-dimensional terms by the formula

$$\frac{R_w}{mg} = \frac{2\beta\text{Fr}_L}{\pi\alpha} \int_{k_0(k)}^{\infty} |I(k)|^2 \frac{(\sinh kh - \sinh k(h - 1/\beta))^2}{k \cosh^2 kh \sqrt{\text{Fr}_L^2 k^2 - k \tanh kh}} dk, \quad (2.4)$$

where the function $I(k)$ is given by

$$I(k) = \int_{-1/2}^{1/2} f'(\hat{x}) e^{i\hat{x}\sqrt{k/\text{Fr}_L^2 \tanh kh}} d\hat{x}, \quad (2.5)$$

and $\hat{x} = x/L$. The minimum non-dimensional wave number in the integral (2.4), $k_0(k)$, is given by the formula

$$\text{Fr}_L^2 k_0(k) - \tanh kh = 0. \quad (2.6)$$

Note that Sretensky's formula is not capable of predicting skin and form drag since it neglects viscosity. However, we can measure these using a simple wind tunnel experiment and add them (as in (2.3)) to the theoretical prediction of the wave drag. As discussed by Benham *et al.* (2019), this allows us to compare with our tow-tank results. In addition, we note that the skin and form drag do not vary much between the different values of h/L that we study, and so it is acceptable to assume the same values for all depths.

Unlike Sretensky's formula which is written in terms of the hull shape $f(\hat{x})$, the theory of Havelock (1922) relies on knowledge of the pressure disturbance caused by the hull instead. Since it is difficult to know the pressure disturbance exactly, it is often assumed to be a Gaussian distribution with the appropriate aspect ratio values. Hence, we take the surface pressure distribution to be

$$p(\hat{x}, \hat{y})/P_0 = e^{-(\hat{x}^2/2 + \alpha^2 \hat{y}^2/2)}, \quad (2.7)$$

where the magnitude of the pressure distribution P_0 is unknown, but should be of the same order of magnitude as the pressure needed to support a surface disturbance of height comparable to the draft. This can be confirmed by our observation of typical wave amplitudes compared to the total draft. Hence, we take $P_0 = \kappa\rho gH$, where the choice of κ is discussed later. Given this pressure field, Havelock's formula (after simplification) gives the drag as

$$\frac{R_w}{mg} = \frac{\kappa^2}{4\text{Fr}_L^2 \alpha \beta} \int_0^{\infty} \frac{a(k^*(k_y), k_y)}{\partial b / \partial k_x(k^*(k_y), k_y)} dk_y, \quad (2.8)$$

where the functions a and b are given by

$$a(k_x, k_y) = e^{-\frac{1}{2\pi^2}(k_x^2 + k_y^2/\alpha^2)} k \tanh kh, \quad (2.9)$$

$$b(k_x, k_y) = (k \tanh kh) / (k_x \text{Fr}_L^2) - k_x, \quad (2.10)$$

and the wavenumber magnitude is $k = \sqrt{k_x^2 + k_y^2}$. Finally, the critical wavenumber $k^*(k_y)$ is given by

$$b(k^*(k_y), k_y) = 0. \quad (2.11)$$

A recent study by Benham *et al.* (2019) showed that the wave drag at infinite depth can be well-approximated by a modification to the theory of Michell (1898) (the precursor to that of Sretensky), where the shape $f(\hat{x})$ is replaced by the combined shape of the hull and its surrounding boundary layer profile $f(\hat{x}) + \delta(\hat{x})$. In particular, the presence of the

boundary layer distinguishes the difference between forward and backward motion for an asymmetric hull, which Michell’s formula alone is incapable of predicting. Though we focus on symmetric hulls here, we nevertheless make the same modification to Sretensky’s formula (2.4) by including the boundary layer profile, which we extract from a steady k - ω Shear Stress Transport (Menter 1994) numerical simulation (see (Benham *et al.* 2019) for more details). Our calculated boundary layer profile is illustrated in Figure 2(d). This modification appears to give better comparison with experimental results than the original.

Theoretical predictions for the wave drag are compared to the experimental results in Figure 2 (a,b,c). Qualitatively, all the theoretical approaches capture the drag behaviour, though with varying degrees of accuracy. We find that the best comparison is with Havelock’s formula, where the magnitude of the pressure disturbance (2.7) is fitted to give $\kappa = 0.4$, with mean relative error 8%. The next best comparison is our modified version of Sretensky’s formula (with mean relative error 9%), and finally Sretensky’s original formula (with mean relative error 13%). However, we note that the best accuracy near the critical regime $Fr_h \sim 1$ is exhibited by our modified version of Sretensky’s formula, since Havelock’s formula under-predicts drag here.

We note that linear theory can only be expected to perform well when the wave magnitude is small, and indeed considerably smaller than the water depth $A/h \ll 1$. Otherwise, nonlinear effects are expected to have an important role, such as wave steepening, cresting, breaking, and even wetting the subsurface. However, we restrict our experimental study to the range $h/L > 0.24$, and the maximum observed wave amplitude we observe has $A/h \approx 0.25$. Therefore, we conclude that nonlinear effects bear little importance here. Furthermore, we ignore capillary effects since they do not present a significant contribution to the drag (Benham *et al.* 2019).

Each of the theoretical predictions confirm that multiple solutions exist near the vicinity of $Fr_h \sim 1$, though the precise shape of the drag curve differs. Since our modified version of Sretensky’s formula shows the best comparison with experimental data near the critical regime, we choose this theory to provide a means of predicting the drag in between our experimentally measured points. In this way, we can accurately describe the motion of a boat with constant forcing using the quasi-steady system (2.1)-(2.2).

3. Bifurcation diagrams and stability criteria

There are two bifurcation parameters which govern the nonlinear dynamics of the system (2.1)-(2.2): the depth h/L and the forcing F/mg . We display the bifurcation diagrams for each of these parameters in Figure 3, illustrating the different available equilibrium speeds (represented by Fr_L). To explore the bifurcation parameter F/mg , we fix the depth at $h/L = 0.25$. In this case, three solutions exist in the range $F/mg \in (0.07, 0.095)$. To explore the bifurcation parameter h/L , we fix the forcing at $F/mg = 0.08$. In this case, three solutions exist in the range $h/L \in (1/\beta, 0.4)$ (No solution exists when the depth is shallower than the boat itself $h/L \leq 1/\beta$).

The different solution branches may correspond to significantly different Froude numbers, and hence equilibrium speeds. For example, at forcing $F/mg = 0.08$, the available Froude numbers are between 0.4 and 0.7. That is to say, a boat which is pushed at this forcing could be going at three possible speeds, where the fastest speed is almost double the slowest. Evidently, this has significant implications for rowing and sailing races in waters with depth varying between deep and shallow regions, since jumping from one branch to another could result in hugely improved race times, and possibly even new world records.

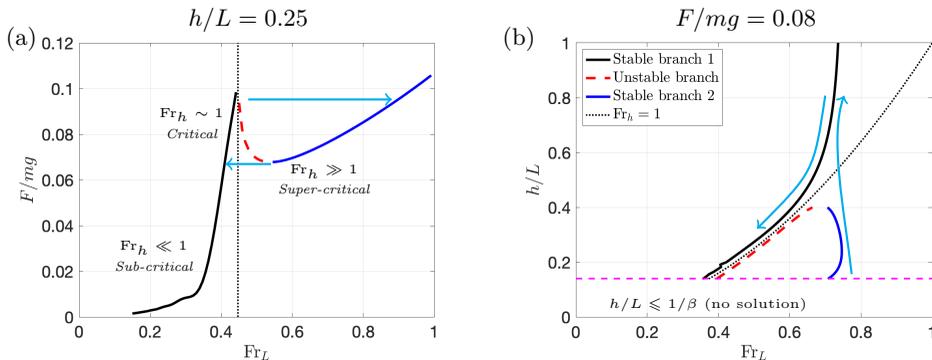


Figure 3: Bifurcation diagrams, showing all of the existing branches of the quasi-steady equilibrium solutions to (2.1)-(2.2), for the two bifurcation parameters F/mg and h/L . We also illustrate the different sub-critical, critical and super-critical regimes, as well as possible hysteresis routes.

To assess the stability of these different branches, we now perform a linear analysis. Consider an equilibrium solution (x_0, u_0) , such that the right hand side of (2.1)-(2.2) is zero. Then consider a small perturbation away from this solution

$$x = x_0 + \epsilon(t), \quad u = u_0 + \dot{\epsilon}(t). \quad (3.1)$$

For the simplicity of the stability analysis, we restrict our attention to the case where the depth is constant (such that $\partial R_d / \partial x = 0$). Then, inserting (3.1) into the system (2.1)-(2.2), expanding in terms of the small variable $\epsilon(t)/L \ll 1$, and keeping only leading order terms, we find that

$$(m + m_a)\ddot{\epsilon} + \frac{\partial R_d}{\partial u}(x_0, u_0)\dot{\epsilon} = 0. \quad (3.2)$$

In this case, the stability is purely determined by the sign of $\partial R_d / \partial u$, or equivalently $\partial R_d / \partial Fr_L$. Consequently, all solutions in Figure 3(a) which lie on the part of the drag curve with negative slope are unstable, and all the solutions which lie on the positive slope are stable. Using this knowledge, we have illustrated the corresponding stability of the different branches in the bifurcation diagrams using dashed lines. Likewise, the corresponding unstable branch in Figure 3(a) is also indicated.

We also display several possible hysteresis routes during motion. For example, in (a) we show how slowly increasing the forcing (starting from zero) would cause a jump increase in velocity at the turning point near $Fr_h = 1$. From here, slowly decreasing the velocity would return on a different solution branch, until a jump back to the original branch at the second turning point. Similarly, we show a possible hysteresis route in (b) by varying the depth instead.

4. It came from the deep...

Having validated the theoretical results against experimental data, and having explored the different available solution branches that occur at various depths and forcing, now we apply the system (2.1)-(2.2) to a case study. Using our modified version of Sretensky's theory (for the wave drag), and wind tunnel measurements (for the form and skin drag), we investigate the quasi-steady dynamics of a body moving in a lake of length nL with

a sinusoidal depth profile of the form

$$h(x) = 1/2[(h_{\max} + h_{\min}) - (h_{\max} - h_{\min}) \cos 2\pi x/nL]. \quad (4.1)$$

We choose a large (but realistic) race length $n = 400$ to ensure a slowly varying depth, and we set the maximum and minimum depths as $h_{\max}/L = 1$ and $h_{\min}/L = 0.2$ to ensure we do not venture outside of the region we have studied experimentally (so that our predictions remain accurate). With these values, the maximum calculated value for the acceleration due to depth change is $\dot{u} \approx 0.6 \text{ m/s}^2$, which is much smaller than $g = 9.8 \text{ m/s}^2$, indicating that the quasi-steady approximation is appropriate.

Motion from shallow to deep water corresponds to $0 < x < nL/2$, and likewise from deep to shallow water $nL/2 < x < nL$. This canonical depth profile not only exhibits the nonlinear phenomena we have illustrated both experimentally and theoretically, but also is a physically realistic scenario. There are many lakes and rivers that have patches of shallow water, and this study illustrates the possible outcomes that can occur as a boat passes over such a patch.

We consider a sequence of forcing magnitudes F_i/mg between 0.005 and 0.11, and investigate the quasi-steady dynamics in each of these cases. Such forcing values cover the range relevant for rowing races, where typical forces are around 6 – 8% of the total boat weight (Boucher *et al.* 2018). At $x = 0$, where the depth is at its minimum value, two possible stable solutions are available for all forcing values in the range $0.07 < F/mg < 0.11$. Therefore, for each forcing value i we choose initial conditions for the system (2.1)-(2.2) as one of these two branches

$$x_{i,j}(0) = 0, \quad u_{i,j}(0) = u_{i,j}^*, \quad j = 1, 2, \quad (4.2)$$

where the subscript j indicates the branch number, and the starting velocity $u_{i,j}^*$ is one of the two stable solutions to the equilibrium equation

$$R_d(0, u_i^*) = F_i. \quad (4.3)$$

Then, we solve the system (2.1)-(2.2) for $0 < t < T$, where the final time is determined by the condition $x(T) = nL$. We plot various solutions in Figure 4, where we indicate different stable branches with different colours. Figure 4(a) shows the solutions on the phase plane generated by $h(t)/L$ and $\text{Fr}_L(t)$. We plot these on top a colour map of the non-dimensional wave drag coefficient $C_w = R_w/\rho u^2(LHW)^{2/3}$, to illustrate pronounced regions of drag more clearly. Figure 4(b) shows the solutions $\text{Fr}_L(t)$ plotted against $x(t)/nL$.

The behaviour is remarkably different depending on the forcing magnitude. If the forcing is less than a critical value $F_c/mg = 0.07$, then only one solution branch exists for all values of x . The drag is largest in the shallow part and smallest in the deep part, with perfectly symmetric motion about $x = nL/2$. On the other hand, for $F > F_c$ the catastrophic emergence of multiple solution branches and a hysteresis pattern is observed. One branch is a very slow solution, forced beneath a drag barrier at $\text{Fr}_L = \sqrt{h/L}$ (or $\text{Fr}_h = 1$) that is visible as a bright yellow patch in the drag coefficient in 4(a), and behaves much like the solutions observed for small forcing. The other stable branch of the solution is much faster - in fact, even faster than the equilibrium solution at infinite depth. However, on the return journey both branches collapse to the same trajectory due to the fact that the fold bifurcation is topologically disconnected. Nevertheless, boats which stay on this stable branch have a significantly better race time than the former ones.

We plot the race times $T_{1/2}$ for the first half of the race $0 \leq x \leq nL/2$, for various values of the forcing in 4(c). We give the times in terms of a dimensional scaling for the

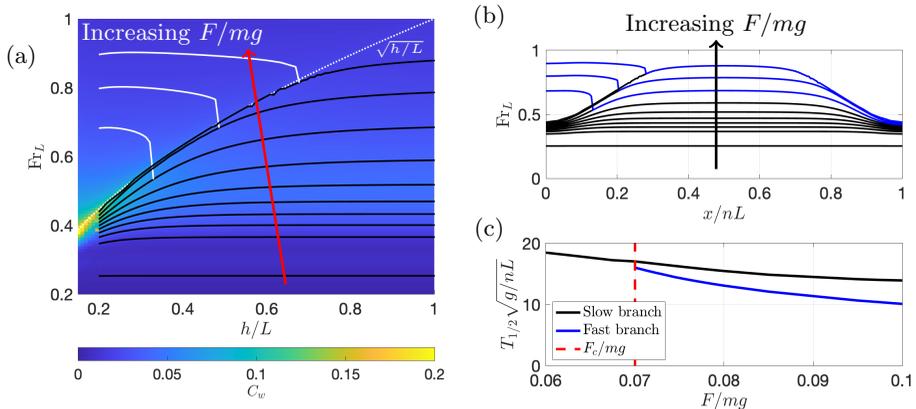


Figure 4: Phase plane analysis of the case study of a boat moving in a lake with sinusoidal depth (4.1), where the rate of change of depth is sufficiently small that the quasi-steady approximation is appropriate. Trajectories for different values of the forcing term F/mg are illustrated in both the depth-Froude plane (a) and the distance-Froude plane (b). (c) Times to complete the first half of the race $0 \leq x \leq nL/2$ for the different solution branches.

total race $\sqrt{nL/g}$, and we illustrate where the two stable branches split at the critical forcing. We ignore the second half of the race, since both branches converge to the same path. Well above the critical forcing magnitude F_c/mg , race times between branches can differ by as much as 30%. This critical value is important because force values $F > F_c$ present an opportunity to beat records in racing applications.

5. Discussion and perspectives

We have investigated the quasi-steady dynamics of a body moving near the surface of a region of water with slowly varying depth. This work sheds light on previously unstudied bifurcation behaviour that emerges in shallow water. In particular, the two key parameters are the depth h/L and the forcing F/mg , which can produce as many as three different solution branches (two stable and one unstable) with potentially large differences in speeds. We showed how this can produce hysteresis routes, and can have significant impact on the time to complete a race in a course with varying depth. In particular, there exists a critical forcing value F_c above which there is an opportunity to accelerate as one enters a shallow patch of water, or be quashed by a rising drag barrier, depending on one's particular branch of the solution.

There are several interesting perspectives following on from this work. Firstly, in boat sports, a comprehensive study of the effect of race bathymetry on performance for different athletes could be investigated and compared with our results. It would also be interesting to study the interactions between multiple bodies moving in proximity, and how this affects the results presented here.

Secondly, the extension of this work to stratified flow could be investigated. In particular, for very sharp stable stratification with heavier water below lighter water (such as in a fresh water glacial run-off into the sea) it is anticipated that the heavier water would act approximately like a solid barrier at finite depth. Hence, it is possible that similar effects to those discussed here would be observed. If this is the case, then our

results could shed light on the famous dead water problem (Ekman 1904), where drag is enhanced in sharp stable stratification. It would be interesting to see if there exists an *anti-dead* (or *alive*) water regime, where the drag diminishes on a different solution branch.

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