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Wave resistance for capillary gravity waves: Finite-size effects

M. Benzaquen[1], F. Chevy[2] and E. Raphaël[1(a)]

[1] *Laboratoire PCT - UMR Gulliver CNRS 7083, ESPCI - 10 rue Vauquelin, 75005 Paris, France, EU*
[2] *Laboratoire Kastler Brossel, ENS, Université Paris 6, CNRS - 24 rue Lhomond, 75005 Paris, France, EU*

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Abstract – We study theoretically the capillary-gravity waves created at the water-air interface by an external surface pressure distribution symmetrical about a point and moving at constant velocity along a linear trajectory. Within the framework of linear wave theory and assuming the fluid to be inviscid, we calculate the wave resistance experienced by the perturbation as a function of its size (compared to the capillary length). In particular, we analyze how the amplitude of the jump occurring at the minimum phase speed $c_{\min} = (4\, g\gamma/\rho)^{1/4}$ depends on the size of the pressure distribution (ρ is the liquid density, γ is the water-air surface tension, and g is the acceleration due to gravity). We also show how for pressure distributions broader than a few capillary lengths, the result obtained by Havelock for the wave resistance in the particular case of pure gravity waves (*i.e.*, $\gamma = 0$) is progressively recovered.

Introduction. – Water waves are both captivating and of great practical significance [1–3]. They have thus attracted the attention of scientists and engineers for many decades [4]. Water waves can, for instance, be generated by the wind blowing over the ocean, by a moving ship on a calm lake, or simply by throwing a stone into a pond. Their propagation at the surface of water is driven by a balance between the liquid inertia and its tendency, under the action of gravity or of surface tension (or a combination of both in the case of *capillary-gravity waves*), to return to a state of stable equilibrium [5]. The dispersive nature of water waves is responsible for the complicated wave pattern generated at the free surface of a still liquid by a moving disturbance such as a partially immersed object (*e.g.* a boat or an insect) or an external surface pressure source [6]. The propagating waves generated by the moving disturbance continuously remove energy to infinity. Consequently, the disturbance will experience a drag, R, called the *wave resistance*. In the case of ships (for which surface tension is negligible), this drag is known to be a major source of resistance [7] and has been analyzed in detail by Havelock [8]. The case of objects that are small compared to the capillary length has been considered only recently [9–16] and has attracted strong interest in the context of insect locomotion on water surfaces [17–19]. For such objects, one has to take into account both gravity and surface tension. In the case of a point-like surface pressure distribution, this leads to a discontinuity of the wave resistance at a critical velocity given by the minimum of the wave velocity $c_{\min} = (4g\rho/\gamma)^{1/4}$ [9]. For clean water at room temperature, one has $c_{\min} \approx 0.23\,\mathrm{m\,s^{-1}}$. When the velocity V of the surface pressure distribution is smaller than $c_{\min}$, no steady waves are generated and the wave resistance vanishes. Emission of steady waves becomes possible only when $V > c_{\min}$, leading to the onset of a finite wave drag [1]. The wave resistance discontinuity at the critical velocity $c_{\min}$ has been experimentally investigated by several groups [13,14].

It is important to notice that while the analysis of [9] was mainly concerned with a point-like surface pressure distribution[2], real perturbations (like insects) have finite sizes. The aim of the present paper is thus to analyze in detail the role played by the finite size of the surface pressure distribution for the wave resistance[3]. We will see in particular that for b much larger than κ^{-1}, the influence of surface tension is negligible. In that case, the results of Havelock [8] for pure gravity waves are recovered.

(a) E-mail: elie.raphael@espci.fr

[1] As shown in [15,16], the threshold at $V = c_{\min}$ exists only for objects moving with no acceleration.

[2] Note that the particular case where the typical size b is much smaller than the capillary length $\kappa^{-1} = (\gamma/\rho g)^{1/2}$ has also been (briefly) discussed in [9]. It has been shown that the wave resistance displays a maximum at a velocity of the order of $\sqrt{\gamma/(\rho b)}$ [9]. However, the consequences of a pressure distribution of finite size on the amplitude of the discontinuity of the wave resistance at $V = c_{\min}$ has been overlooked in [9].

[3] Note that the effect of a pressure distribution of finite size on the amplitude of the discontinuity of the wave resistance was already briefly discussed in [14].

Formulation. – Consider an incompressible, inviscid, infinitely deep liquid whose free surface is unlimited. We take the xy-plane as the equilibrium surface of the fluid and the z-axis along the upward direction perpendicular to the equilibrium surface. We study the wave motion created by an external surface pressure distribution that moves with speed V in the negative x-direction. In the frame of that moving disturbance, physical quantities are stationary: the pressure distribution is given by $P(x,y)$ and the displacement ζ of the free surface from its equilibrium position is of the form $\zeta(x,y)$.

In order to calculate the wave resistance experienced by the disturbance, we use a method first introduced by Havelock [8]. According to this author, we may imagine a rigid cover fitting the surface everywhere. The assigned pressure system $P(x,y)$ is applied to the liquid by means of this cover; hence the wave resistance is simply the total resolved pressure in the x direction. This leads to

$$R = \int \mathrm{d}x\, \mathrm{d}y\, P(x,y)\partial_x \zeta(x,y). \tag{1}$$

For the sake of simplicity, let us restrict ourselves to the case of a pressure system symmetrical around the origin so that $P(x,y) = g(r)$ with $r = (x^2+y^2)^{1/2}$. The Fourier transform $\hat{P}(k_x,k_y)$ is then a function only of k and can be written as $\hat{P}(k_x,k_y) = G(k)$, where

$$G(k) = \int_0^\infty \mathrm{d}r\, r g(r) \int_0^{2\pi} \mathrm{d}\theta\, e^{-ikr\cos\theta} \tag{2}$$

$$= 2\pi \int_0^\infty \mathrm{d}r\, r g(r) J_0(kr). \tag{3}$$

J_0 denotes the Bessel function of the first kind of zeroth order. It has been shown by Raphaël and De Gennes in [9] that in such a case the wave resistance R (for $V \geqslant c_{\min}$) reduces to

$$R = \int_0^\chi \frac{\mathrm{d}\theta \cos\theta}{\pi\gamma} \frac{\{k_2(\theta)G[k_2(\theta)]\}^2 + \{k_1(\theta)G[k_1(\theta)]\}^2}{k_2(\theta)-k_1(\theta)}, \tag{4}$$

where

$$k_1(\theta) = \kappa \left(\frac{V}{c_{\min}}\right)^2 \{\cos^2\theta - (\cos^4\theta - \cos^4\chi)^{1/2}\},$$
$$k_2(\theta) = \kappa \left(\frac{V}{c_{\min}}\right)^2 \{\cos^2\theta + (\cos^4\theta - \cos^4\chi)^{1/2}\}, \tag{5}$$

where χ is defined by $\cos\chi = c_{\min}/V$. Equation (4) is an important result as it predicts how the wave resistance varies as a function of the velocity for any pressure distribution.

Finite-size effects on the wave resistance. – Equation (4) was studied in detail in [9] in the particular case of a point-like pressure distribution $g(r) = p\,\delta(r)$. We here consider the case of a pressure distribution of finite size, b. For instance, we can assume the pressure distribution to be Gaussian

$$g(r) = \frac{p}{2\pi b^2} \exp\left(-\frac{r^2}{2b^2}\right), \tag{6}$$

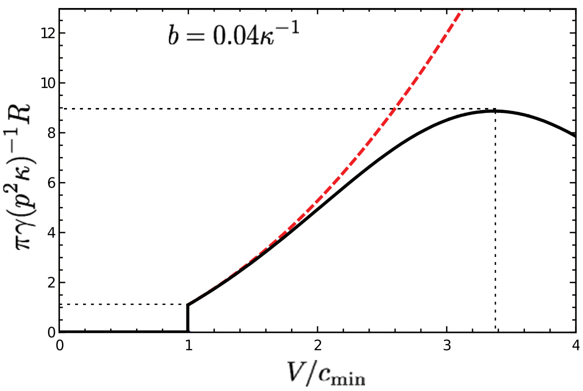

Fig. 1: (Color online) Plot of the wave resistance R in units of $p^2\kappa(\pi\gamma)^{-1}$, as a function of $V/c_{\min}$. The red (dashed) curve corresponds to a point-like pressure source $g(r) = p\,\delta(r)$, the black one (solid line) corresponds to a Gaussian pressure field (6) of size $b = 0.04\,\kappa^{-1}$.

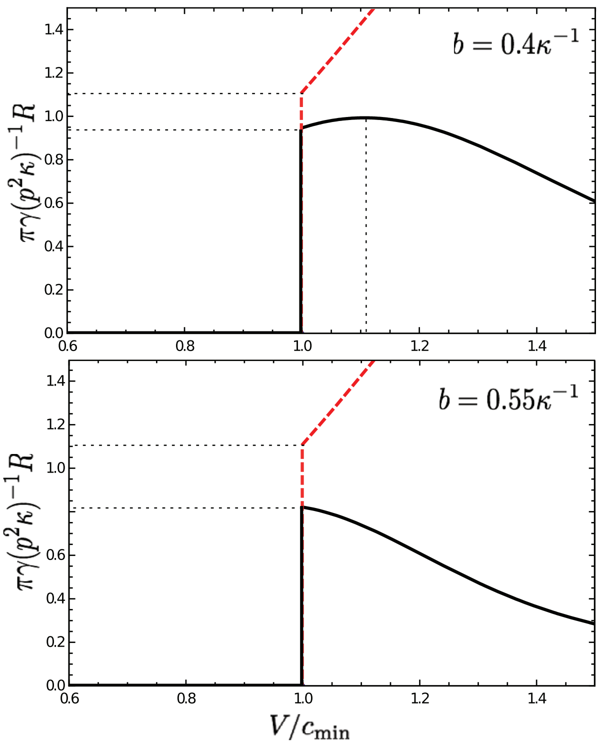

Fig. 2: (Color online) Plot of the wave resistance R in units of $p^2\kappa(\pi\gamma)^{-1}$, as a function of $V/c_{\min}$. The red (dashed) curve corresponds to a point-like pressure field $g(r) = p\,\delta(r)$, the black one (solid line) corresponds to a gaussian pressure field (6) of size $b = 0.4\,\kappa^{-1}$ (upper graph) and $b = 0.55\,\kappa^{-1}$ (lower graph).

Equation (3) then becomes

$$G(k) = p \exp\left(-\frac{b^2k^2}{2}\right). \tag{7}$$

By inserting the Fourier transform of the pressure field $G(k)$ in eq. (4), we obtain the wave resistance as a function of $V/c_{\min}$ (see figs. 1 and 2).

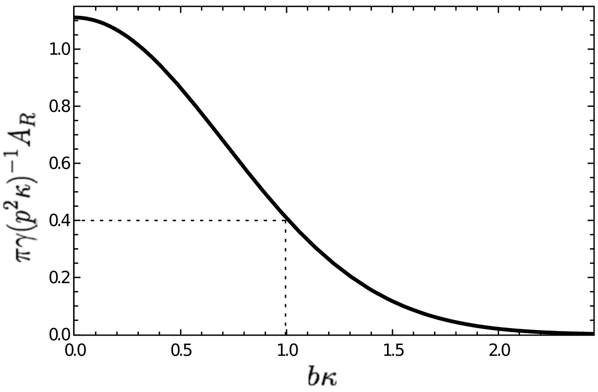

Fig. 3: Plot of the amplitude of the jump A_R of the wave resistance in units of $p^2\kappa(\pi\gamma)^{-1}$, as a function of $b\kappa$.

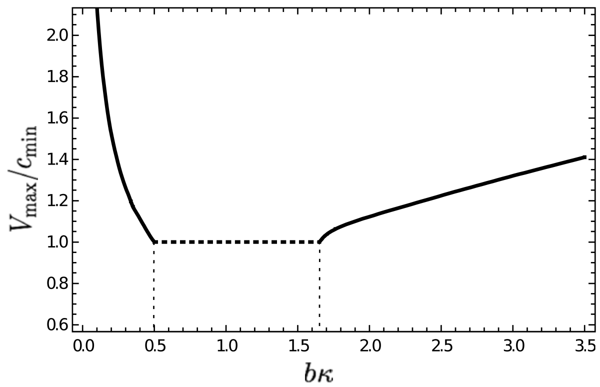

Fig. 4: Plot of $V_{\max}$ corresponding to the maximum of wave resistance in units of $V/c_{\min}$, as a function of $b\kappa$. The black dotted line (horizontal) corresponds to a situation in which the maximum value is reached at $V_{\max}=c_{\min}$. For $b\kappa \ll 1$, $V_{\max}$ scales as $\sqrt{\gamma/(\rho b)}$, whether for $b\kappa \gg 1$, $V_{\max}$ scales as $\sqrt{gb}$.

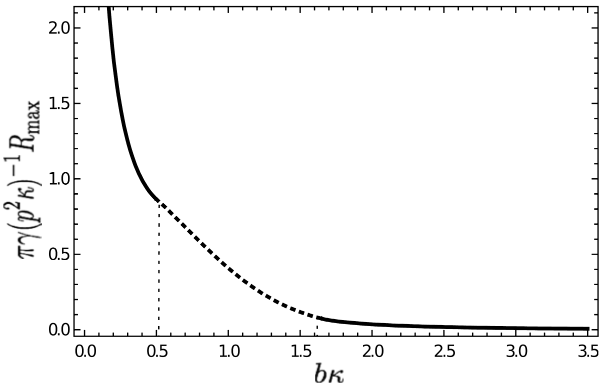

Fig. 5: Plot of the maximum of wave resistance $R_{\max}$ in units of $p^2\kappa(\pi\gamma)^{-1}$, as a function of $b\kappa$. The dotted line corresponds to a situation in which the maximum value is reached at $V_{\max}=c_{\min}$.

The red (dashed) in curve in fig. 1 corresponds to a point-like pressure distribution $g(r)=p\,\delta(r)$, the black one (solid line) corresponds to a Gaussian pressure field (6) of size $b=0.04\,\kappa^{-1}$. We clearly observe, for both curves, the discontinuity (or jump) at $V=c_{\min}$ that we discussed earlier. In fig. 1, the black curve (solid line) presents a maximum at $V_{\max}\sim\sqrt{\gamma/(\rho b)}$ which is the first consequence of the finite-size effects. This separates the behavior of the wave resistance in two regimes: below $V_{\max}$, R increases with the disturbance velocity whereas for $V>V_{\max}$, R decreases with V.

In fig. 1, the disturbance typical size b is much smaller than the capillary length κ^{-1}. When b becomes of the same order of magnitude than κ^{-1} (see fig. 2), the amplitude of the jump, denoted by A_R in the following, decreases as shown in fig. 3. When the typical size b gets close enough to the capillary length κ^{-1}, the maximum value of the wave resistance is obtained for $V_{\max}=c_{\min}$. Such a situation is depicted in the lower graph in fig. 2. Figures 4 and 5 give quantitative information on this situation: we show that in the particular case of a Gaussian pressure field it occurs for $0.5\,\kappa^{-1}\lesssim b\lesssim 1.65\,\kappa^{-1}$. Note that the above interval is only valid for the Gaussian pressure field (6). If, for instance, one uses a Lorentzian pressure source $p(2\pi)^{-1}b(b^2+r^2)^{-3/2}$ instead of (6), the above interval becomes $0.3\,\kappa^{-1}\lesssim b\lesssim 2.2\,\kappa^{-1}$.

From eq. (4) on can get the jump amplitude at $V=c_{\min}$:

$$A_R=\frac{\kappa}{2\sqrt{2}}\frac{G(\kappa)^2}{\gamma}=\left(\frac{p^2}{\pi\gamma}\kappa\right)\frac{\pi}{2\sqrt{2}}\left(\frac{G(\kappa)}{p}\right)^2. \qquad (8)$$

Equation (8) is an important result as it gives how the jump of the wave resistance at the critical velocity $c_{\min}$ varies with the size of the pressure distribution. In the particular case of the Gaussian pressure field given in eq. (6), the jump amplitude becomes

$$A_R=\left(\frac{p^2}{\pi\gamma}\kappa\right)\frac{\pi}{2\sqrt{2}}\,\exp\left(-b^2k^2\right). \qquad (9)$$

Note that in the limit $b\to 0$, the result of [9] $A_R=\pi/(2\sqrt{2})$ for of a point-like pressure distribution is recovered. The amplitude of the jump A_R of the wave resistance in units of $p^2\kappa(\pi\gamma)^{-1}$, is depicted in fig. 3 as a function of $b\kappa$. We notably observe that A_R is significantly suppressed when the typical size b becomes greater than $2.5\,\kappa^{-1}$. In such a case, the jump in the wave resistance might be difficult to observe experimentally (see also fig. 6 below).

One may also wonder how the maximum in the wave resistance scales with b. The abscissa ($V_{\max}$) and ordinate ($R_{\max}$) of the maximum of wave resistance as a function of $b\kappa$ are depicted in fig. 4 and fig. 5.

In both figures the dotted line corresponds to the situation in which the maximum value is reached at $V_{\max}=c_{\min}$ (see lower graph in fig. 2).

In the limit $b\kappa\gg 1$, the asymptotic behavior $V_{\max}\simeq\sqrt{gb}$ can be simply recovered by substituting b^{-1} to k in the pure gravity wave dispersion relation $\omega(k)/k=\sqrt{g/k}$. In the opposite limit $b\kappa\ll 1$, the asymptotic behavior $V_{\max}\simeq\sqrt{\gamma/(\rho b)}$ can analogously be obtained by substituting

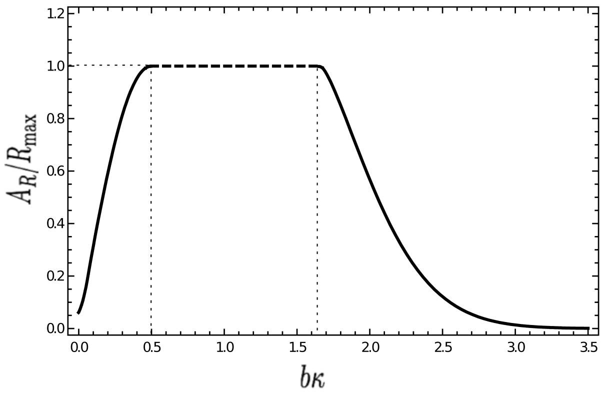

Fig. 6: Plot of the quotient $A_R/R_{\max}$ as a function of $b\kappa$. The dotted line corresponds to a situation in which the maximum value is reached at $V_{\max}=c_{\min}$.

b^{-1} to k in the pure capillary wave dispersion relation $\omega(k)/k=\sqrt{\gamma k/\rho}$ (see footnote [4]).

We observe in fig. 5 that $R_{\max}$ (and hence the whole wave resistance) strongly decreases with the size b of the pressure source. Although this might be surprising at first sight, one has to notice that we have kept constant the magnitude $p=\int \mathrm{d}x\,\mathrm{d}y\,P(x,y)$ of the pressure source while varying b.

In order to discuss the general shape of the graph of the wave resistance, and whether or not the jump in the wave resistance can be easily detected, we plot the ratio between the amplitude A_R and the maximum of wave resistance $R_{\max}$ (see fig. 6). One can see how the ratio $A_R/R_{\max}$ starts at zero (for $\kappa^{-1}\to 0$, A_R is finite and $R_{\max}\to\infty$) and increases until reaching the constant value 1 in the range $0.5\,\kappa^{-1}\lesssim b\lesssim 1.65\,\kappa^{-1}$ discussed earlier (see fig. 2). For larger values of b, the ratio $A_R/R_{\max}$ is essentially suppressed.

Pure gravity waves. – As we have seen above, $V_{\max}$ scales as $\sqrt{gb}$ for $b\kappa\gg 1$. One shall thus wonder if, more generally, our results for the wave resistance are for $b\gg\kappa^{-1}$ well approximated by the result of Havelock [8] for pure gravity waves ($\gamma=0$).

If $\gamma=0$, the wave resistance (4) reduces to

$$R=\frac{g^2}{\rho V^6}\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\cos^5\theta}\left(G\left(\frac{g}{V^2\cos^2\theta}\right)\right)^2, \qquad (10)$$

which, for the Gaussian pressure field (6), yields

$$R=\frac{p^2}{\pi\rho g d^3}\left(\frac{\sqrt{gb}}{V}\right)^6 \times\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\cos^5\theta}\exp\left(-\frac{1}{\cos^4\theta}\left(\frac{\sqrt{gb}}{V}\right)^4\right). \qquad (11)$$

Using eq. (11), one can plot the wave resistance for pure gravity waves as a function of $V/c_{\min}$ for different values of b (see the black curves (solid line) in fig. 7).

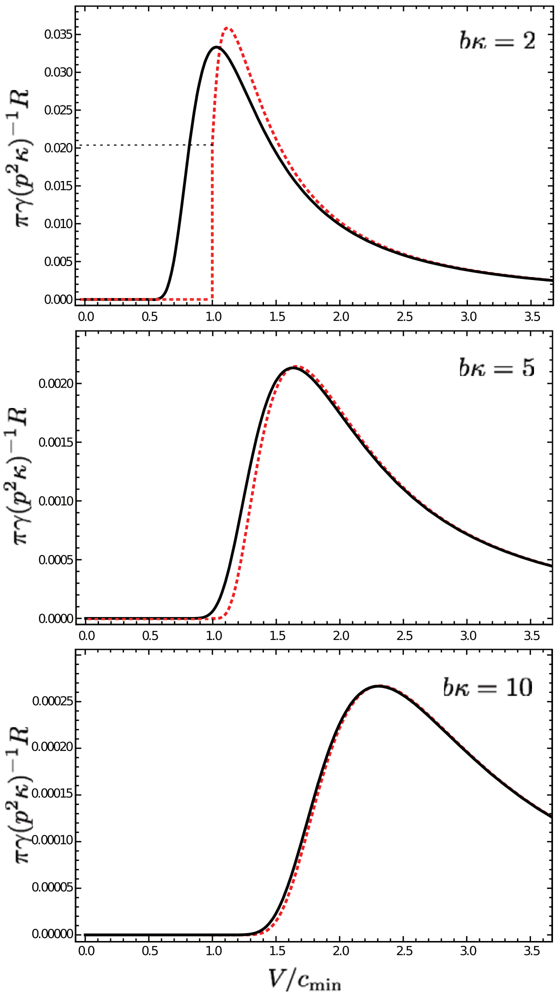

Fig. 7: (Color online) Plot of the wave resistance R in units of $p^2\kappa(\pi\gamma)^{-1}$, as a function of $V/c_{\min}$. The black curve (solid line) corresponds to the formula for pure gravity waves (11), the red one (dashed) corresponds to the initial formula for capillary gravity waves (4). On the top we have $b=2\,\kappa^{-1}$, in the middle we have $b=5\,\kappa^{-1}$, while at the bottom we have $b=10\,\kappa^{-1}$.

Note that $V_{\max}$ is now exactly given by $\sqrt{gb}$. For velocities smaller than $0.6\sqrt{gb}$, the wave resistance is practically unnoticeable[5].

We have also plotted in fig. 7 the wave resistance for capillary-gravity waves (red (dashed) curves).

One can see how both curves come closer to each other as b is increased. For b much larger than κ^{-1}, one does not see much differences between the two curves, meaning that in this limit the problem is essentially ruled by pure gravity waves theory.

Concluding remarks. – We have shown theoretically how the finite size of an external axisymmetric pressure source significantly modify the wave resistance, and in particular its singular behavior at the minimum phase speed $c_{\min}$, compared to the case of a point-like

[4]Note however, that $V_{\max}$ is not simply given by $\sqrt{\gamma/(\rho b)+gb}$.

[5]For velocities up to $0.8\sqrt{gb}$, on can show that eq. (11) is well approximated by $p^2\kappa(\pi\gamma^{-1})\sqrt{\pi/2}\exp(-1/x^4)/(2x^4)$ with $x=V/\sqrt{gb}$.

disturbance[6]. Our study also provides a quantitative description of the crossover between capillary-gravity and purely gravity wave resistance described in previous works.

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[6]In the context of animal locomotion on water surfaces, it would be interesting to generalize the results of the present study to non-axisymmetric pressure sources.