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Trading lightly

Cross-impact and optimal portfolio execution

Trading lightly: cross-impact and optimal portfolio execution

Iacopo Mastromatteo, Michael Benzaquen, Zoltan Eisler and Jean-Philippe Bouchaud model the liquidation costs of a basket of correlated instruments by generalising the linear propagator model previously used for single instruments, obtaining an arbitrage-free cost model. They illustrate their results using a pool of US stocks, showing that neglecting cross-impact effects leads to an incorrect estimation of liquidity and results in suboptimal execution strategies that are not correctly synchronised across different stocks

Executing trading decisions in real markets is a difficult business. Moving around substantial amounts is often foiled by a lack of liquidity. When orders exceed the small volumes typically available at the best bid or ask in lit order books, the incurred slippage costs rapidly become detrimental to the trading strategy. Accurately predicting these trading costs is a non-trivial exercise, and one must often resort to statistical models. Most of the complexity of such models arises from price impact, ie, the fact that trades tend to move market prices in their own direction. This effect has recently triggered a large amount of theoretical activity due to its direct relation to supply and demand as well as the abundantly available data on financial transactions. As alluded to above, this interest is not purely academic, since reliable estimates of trading costs are crucial to judge whether one should enter into a position and – if the cost model is accurate enough – attempt to optimise the execution strategy. For example, splitting large orders into a stream of smaller ones over time is a universally accepted way of mitigating transaction cost. However, as always, the devil is in the details, and the precise nature and time scheduling of the orders can make a large difference to the final result.

Optimal execution problems are widespread in the literature, and many different formulas and techniques have been developed to solve them. However, most of these problems are restricted to single asset execution. In Benzaquen *et al* (2017), we showed that even a simple linear model of cross-asset price impact leads to a very rich phenomenology, in line with the results of Wang *et al* (2015) and Wang & Guhr (2016). In the present paper, we provide a practical recipe to optimise the execution of a portfolio of trades, taking into account the cross-impact on the different underlying products within the multivariate framework of Schied *et al* (2010). We will show that proper synchronisation of the legs of the execution schedule is very important. To quantify the slippage incurred by the strategy, we introduce the EigenLiquidity model (ELM). Ths model is directly related to statistical risk factors that have been used for portfolio risk management for several decades. Based on a principal component analysis of the correlation matrix, which provides a practical method to quantify the different kinds of market risk (long the market, sectorial, etc) one can trade while staying within a prescribed budget of transaction cost.

A quadratic cost model for a single stock

To set the stage, let us first discuss the execution of a single company's stock over a trading day that starts at time $t = 0$ and lasts until $t = T$. The total volume we have to trade is V shares, which is obtained over the day by a continuous execution schedule with local speed $v(t)$, normalised as:

$$\int_0^T v(t)\,\mathrm{d}t = V$$

If we were to hold this position, our expected risk, defined as the standard deviation of the daily profit and loss, would be $\mathcal{R} = \sigma V$, where $\sigma = \mathbb{E}[(p_T - p_0)^2]^{1/2}$ is the daily volatility of the stock, expressed in dollars. For convenience, let us define the trading speed in risk units as $q(t) := \sigma v(t)$ and the total risk exchanged over the day as:

$$Q = \int_0^T \mathrm{d}t\,q(t) := \sigma V$$

which naturally equals $\mathcal{R}$.

A standard model for estimating the cost incurred when trading a certain volume was first introduced by Almgren & Chriss (2001). We will consider their framework in a setting where the trader is risk neutral and impact is linear and transient (Alfonsi & Schied 2013; Busseti & Lillo 2012; Gatheral & Schied 2013; Gatheral *et al* 2012). This allows one to express the trading costs as:

$$\mathcal{C} = \iint_0^T \mathrm{d}t\,\mathrm{d}t'\,q(t)G(t-t')q(t') \tag{1}$$

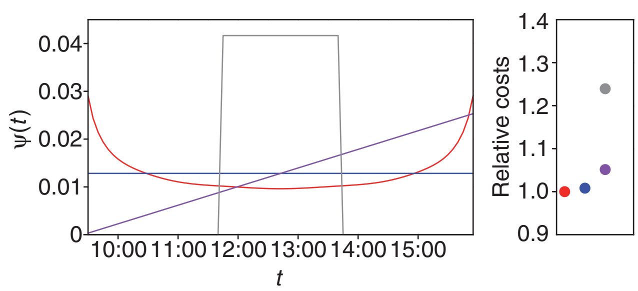
where $G(\tau)$ is an impact kernel (Bouchaud *et al* 2004), which quantifies the effect of a small trade $q(t')\,\mathrm{d}t'$ on the price at a later time $t = t' + \tau$. $G(\tau)$ is typically a decreasing function, dropping from a maximum value obtained at $\tau = 0$ to zero after a slow decay. Consistently with the results of Bouchaud *et al* (2004), it can be written as $G(\tau) = g\phi(\tau)$, with:

$$\phi(\tau) = \begin{cases} (1+\tau/\tau_0)^{-\alpha} & \text{for } \tau \geqslant 0 \\ 0 & \text{for } \tau < 0 \end{cases} \tag{2}$$

$G(\tau)$ has units of 1/\$, and its inverse corresponds to the amount of risk one would have to trade, in the absence of decay, to move the stock's price by its typical daily volatility σ.

In spite of some limitations (eg, impact is empirically found to be a sublinear function of volume), the model above provides a reasonable estimation of trading costs for large trades, and it is able to capture the main effects of the trade schedule on costs (Alfonsi & Schied 2013; Gatheral & Schied 2013; Gatheral *et al* 2012). Extensions of this model to account for risk aversion have been considered in Almgren & Chriss (2001), Obizhaeva & Wang (2013) and Curato *et al* (2016), and for bid-ask spread effects in Obizhaeva & Wang (2013) and Curato *et al* (2016).

1 Typical daily trading profiles for executing one unit of daily risk (left panel)



The red trajectory corresponds to the bucket-shape trajectory minimising the trading costs under the model in (2), with $\alpha = 0.2$ and $\tau_0 = 90$ seconds. The blue trajectory is a flat execution rate throughout the day. The grey trajectory corresponds to a flat rate during two hours at midday, and the purple trajectory is a linear profile, increasing from morning to afternoon. The right panel provides the relative cost of each of the trading trajectories with respect to the optimal one.

Optimal trading of a single stock

Optimal trading schedules under the cost function (1) for a fixed total volume to execute have been extensively investigated in the finance literature (Alfonsi & Schied 2013; Busseti & Lillo 2012; Gatheral & Schied 2013; Gatheral *et al* 2012; Obizhaeva & Wang 2013). The trajectory of minimum cost can be written as $q(t) = Q\psi^{\star}(t)$, where $\int \mathrm{d}t\ \psi^{\star}(t) = 1$ and $\psi^{\star}(t)$ can be determined by solving a linear integral equation (Gatheral *et al* 2012). This yields the well-known symmetric bucket-shape solution for $\psi^{\star}(t)$ depicted in figure 1 (red curve). The optimal solution indicates that after an initial period of faster trading, one should slow down the execution to limit the extra cost due to the effect of one's own trades, and then accelerate the trading again near the market close. Since trading does not extend beyond that point, strongly affecting the price in this final period does not penalise any further executions. As an example, the optimal policy is about 30% less expensive than a localised flat two-hour execution, and approximately 7% cheaper than the linear trading profile represented in figure 1.

The temporal shape of the impact kernel precludes price manipulation, meaning no round-trip trajectory is capable of making money on average (Alfonsi & Schied 2010; Alfonsi *et al* 2012; Gatheral 2010).

A quadratic cost model for portfolios

The problem of optimal execution across multiple instruments was first considered in Schied *et al* (2010), Kratz & Schöneborn (2014) and Schöneborn (2016), whereas the cost model defined above has been generalised to the multivariate case in Alfonsi *et al* (2016) (see also Schneider & Lillo (2016) for a nonlinear generalisation). Within that framework, our definition of risk above is readily extended to a portfolio of multiple stocks with risk positions $\boldsymbol{Q} = \{Q^i\}_{i=1}^N$ as:

$$\mathcal{R}^2 = \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\rho}\boldsymbol{Q} \tag{3}$$

where the correlation matrix $\boldsymbol{\rho}$ is constructed from the price covariance matrix:

$$\boldsymbol{\Sigma} = \mathbb{E}[(\boldsymbol{p}_T - \boldsymbol{p}_0)(\boldsymbol{p}_T - \boldsymbol{p}_0)^{\mathrm{T}}]$$

through:

$$\rho^{ij} = \frac{\Sigma^{ij}}{\sqrt{\Sigma^{ii}\,\Sigma^{jj}}} \tag{4}$$

The daily volatilities are generalised to:

$$\boldsymbol{\sigma} = \{\sigma^i\}_{i=1}^N = \{(\Sigma^{ii})^{1/2}\}_{i=1}^N$$

Similarly, (1) can be extended to this setting as:

$$\mathcal{C} = \iint_0^T \mathrm{d}t\,\mathrm{d}t'\boldsymbol{q}^{\mathrm{T}}(t)\boldsymbol{G}(t-t')\boldsymbol{q}(t') \tag{5}$$

The interpretation of the matrix elements of:

$$\boldsymbol{G}(t-t') = \{G^{ij}(t-t')\}_{ij=1}^N$$

is as before: after trading $\mathrm{d}q^j(t')$ dollars of risk on the contract j at time t', we expect the price of contract i to change by $G^{ij}(t-t')\,\mathrm{d}q^j(t')$ units of its daily dollar volatility σ^i. The terms with $i = j$ correspond to direct price impact, which was already described by earlier models where each stock was independent. In addition, the new terms with $i \neq j$ describe cross-impact between stocks, which, as was shown by Benzaquen *et al* (2017), is a highly relevant effect, since it explains an important fraction of the cross-correlation between stocks. We will use this feature below.

Benzaquen *et al* (2017) have further found that, within a good degree of approximation, one can write the kernel $\boldsymbol{G}(\tau)$ in the factorised form:

$$\boldsymbol{G}(\tau) = [\boldsymbol{G}_+ + \boldsymbol{G}_-]\phi(\tau)$$

with $\phi(\tau)$ given by (2), and $\boldsymbol{G}_\pm$ denoting the symmetric and antisymmetric part of $\boldsymbol{G}$, respectively. Schneider & Lillo (2016) have shown that when the antisymmetric part of the propagator $\boldsymbol{G}_-$ is large, then price manipulation is possible, leading to an ill-defined cost of optimal strategies. Since, empirically, $\boldsymbol{G}_-$ is small, we set it to zero, so the cost associated with the execution of a portfolio of trades reads:

$$\mathcal{C} = \frac{1}{2}\iint_0^T \mathrm{d}t\,\mathrm{d}t'\,(\boldsymbol{q}^{\mathrm{T}}(t)\boldsymbol{G}_+\boldsymbol{q}(t'))\phi(|t-t'|) \tag{6}$$

The condition of symmetry for $\boldsymbol{G}$ is not the only one required in order to avoid price manipulation. In fact, (6) is free of price manipulation if and only if $\boldsymbol{G}_+$ is positive semidefinite. This amounts to saying that buying a portfolio always pushes its price up, and vice versa, regardless of its composition. This results in an impact cost that is always greater than or equal to zero, independently of the portfolio that is actually traded.

The EigenLiquidity model

In principle, $\boldsymbol{G}$ can be determined using simultaneous trades and quotes data for the corresponding pool of stocks. However, the empirical impact matrix $\boldsymbol{G}$ is, in general, extremely noisy, so some cleaning scheme is necessary. By leveraging the empirical results of Benzaquen *et al* (2017) (see figure 7 therein), which show the structure of the impact matrix is in a suitable statistical sense 'close' to that of the correlation matrix, we assume the impact matrix has the same set of eigenvectors as the correlation matrix $\boldsymbol{\rho}$. Intuitively, the eigenvectors of $\boldsymbol{\rho}$ correspond to portfolios with uncorrelated returns. Our assumption means that trading one of these portfolios will only affect (to first approximation) the returns of that portfolio, and not those of any other orthogonal one. Besides being an empirically reasonable cleaning scheme for $\boldsymbol{G}$, our choice is motivated by the results illustrated in this section, which show that this model leads to a cost function $\mathcal{C}$ satisfying three fundamental consistency requirements: symmetry, positive semi-definiteness and fragmentation invariance.

More precisely, one can write:

$$\boldsymbol{\rho} = \boldsymbol{O}\boldsymbol{\Lambda}\boldsymbol{O}^{\mathrm{T}} \tag{7}$$

where $\boldsymbol{O} = \{O^{ia}\}_{ia=1}^{N}$ is an $N \times N$ orthogonal matrix of eigenvectors and $\boldsymbol{\Lambda} = \{\Lambda^a \delta^{ab}\}_{ab=1}^{N}$ is a diagonal matrix of N non-negative eigenvalues. Our assumption is that the matrix $\boldsymbol{G}$ has the following structure:

$$\boldsymbol{G} = \boldsymbol{O}\boldsymbol{\Lambda}\boldsymbol{g}\boldsymbol{O}^{\mathrm{T}} := \boldsymbol{\rho}^{1/2}\boldsymbol{g}(\boldsymbol{\rho}^{1/2})^{\mathrm{T}} \tag{8}$$

where $\boldsymbol{g} = \{g^a \delta^{ab}\}_{ab=1}^{N}$ is a diagonal matrix, and $\boldsymbol{\rho}^{1/2} = \boldsymbol{O}\boldsymbol{\Lambda}^{1/2}$.

An important property of this decomposition is that it leads to a fragmentation invariant cost formula in the following sense. When trading two completely correlated products i and j (ie, when $\rho^{ij} = 1$), the impact of a trading trajectory does not depend on how the volume is split between the two instruments. More formally, a fragmentation invariant cost $\mathcal{C}$ is left unchanged under the transformation:

$$q^i(t) \to q^i(t) + \delta q(t) \tag{9}$$

$$q^j(t) \to q^j(t) - \delta q(t) \tag{10}$$

where $\delta q(t)$ is completely arbitrary. Intuitively, (8) fulfils this property because of the factor $\boldsymbol{\Lambda}$ multiplying $\boldsymbol{g}$. If instruments i and j are completely correlated, then the relative mode 'rel' is an eigenvector of zero risk, with $\Lambda^{\mathrm{rel}} = 0$. When used for estimating the cost of an execution trajectory $\boldsymbol{q}(t)$, (8) will single out such relative modes through the projection $\boldsymbol{O}^{\mathrm{T}}\boldsymbol{q}(t)$, and it will weight them by the corresponding risk $\boldsymbol{\Lambda}$, which is zero.

The impact model (8) will be called the ELM. It is the most natural choice among all the models implementing fragmentation invariance. In fact, it continuously interpolates between small risk modes (which are expected to be characterised by having a small impact) and large risk modes (for which impact costs can be substantial).

Empirically, (8) has been shown to hold to a good degree of approximation (see Benzaquen *et al* 2017). However, the condition of positive semi-definiteness of $\boldsymbol{G}$, ie, $g^a \geqslant 0$ for all a, is not guaranteed from (8) and thus should be checked using empirical data. This is what we display in figure 3. Here, we use real data and confirm that all the g^a are actually strictly positive. The quantity $(g^a)^{-1}$ has the natural interpretation of liquidity per mode. It expresses the amount of daily risk in dollars that one can trade on the eigen-portfolio a to move its price by its daily volatility $\sqrt{\Lambda^a}$.

Optimal trading of portfolios

Under the ELM, the impact cost of any schedule $\boldsymbol{q}(t)$ admits an interpretation in terms of the modes of the correlation matrix of normalised returns through the decomposition:

$$\mathcal{C} = \frac{1}{2}\sum_{a=1}^{N} g^a \|\tilde{q}^a\|^2 \tag{11}$$

where:

$$\|\tilde{q}^a\|^2 = \iint_0^T \mathrm{d}t\,\mathrm{d}t'\,\tilde{q}^a(t)\phi(|t-t'|)\tilde{q}^a(t')$$

We also introduce the notation:

$$\tilde{\boldsymbol{q}}(t) = (\boldsymbol{\rho}^{1/2})^{\mathrm{T}}\boldsymbol{q}(t) \tag{12}$$

denoting the projection of the executed volumes on a set of uncorrelated, unit-risk eigen-portfolios $\boldsymbol{\rho}^{-1/2} = \{\boldsymbol{\pi}^a\}_{a=1}^{N}$.

The notion of eigen-portfolios is very useful for intuitively characterising the cost formula (11). The name comes from the portfolios being uncorrelated and having unit risk (ie, $(\boldsymbol{\pi}^a)^{\mathrm{T}}\boldsymbol{\rho}\boldsymbol{\pi}^b = \delta^{ab}$), and that trading an amount of the basket a according to the weights given by $\boldsymbol{\pi}^a$ has no impact on the total value of the basket b, and vice versa (ie, $(\boldsymbol{\pi}^a)^{\mathrm{T}}\boldsymbol{G}\boldsymbol{\pi}^b = \delta^{ab}g^a$). This is precisely the intuition behind our central assumption (8).

This construction implies the cost $\mathcal{C}$ can be calculated by first projecting the strategy $\boldsymbol{q}(t)$ on the portfolios $\boldsymbol{\pi}^a$ via (12) and then taking the sum of an impact cost per mode g^a with the weighting factor $\|\tilde{q}^a\|^2$ given by such projections.

Equation (11) also shows that the positivity of the matrix $\boldsymbol{g}$ and the kernel $\phi(\tau)$ makes the optimisation problem convex, which always has a unique solution. The optimum under the terminal constraint $\boldsymbol{Q} = \int_0^T \mathrm{d}t\,\boldsymbol{q}(t)$ is necessarily achieved under a synchronous execution schedule, where at any given point in time all stocks are traded with the same time profile, ie, $\boldsymbol{q}(t) = \boldsymbol{Q}\psi^{\star}(t)$, resembling the case without cross-impact.

Intuitively, an asynchronous execution strategy can be seen in the mode space as the optimal one above plus a round trip along some of these modes. The convexity of the cost function (6) implies that round trips always increase execution costs, so they should be avoided.[1] Hence, synchronicity is a general consequence of the convexity of the problem, together with the homogeneity of the decay kernels $\phi(\tau)$ for different instruments.

A toy example to explain the implications of the formalism is given in box A.

Applications to real data

We have fitted the ELM to a pool of 150 US stocks in 2012, following the procedure described in detail in box B. In Benzaquen *et al* (2017), we find that $\alpha \approx 0.15$ and $\tau_0 \approx 90$ seconds for the time decay of impact.

A visual representation of the impact matrix is given in figure 2. The inhomogeneity of $\boldsymbol{G}$ captures the sectorial structure of the market, encoding the specific dependence of stock i on its sector and/or its most correlated stocks j.

The main difference between our ELM and a model without cross-impact is the impact costs of a large buy programme can no longer be reduced as effectively by spreading the orders across multiple correlated instruments. The impact kernel diffuses the interaction across markets and sectors through the modes. The transaction cost of trading Q dollars of risk in the mode $\boldsymbol{\pi}^a$ is equal to $g^a Q^2$ dollars. Figure 3 shows the inverse of the eigenvalues g^a; this is about \$30 million of risk on the most liquid mode. Most of the cross-interaction between stocks is captured by this market mode, which accounts for the fact that when buying a dollar of risk

[1] *One may compare this with a related discussion by Wang (2017) investigating the case of round trips on two stocks.*

A. A toy example with two stocks

A simple realistic case is $N = 2$ stocks and an impact matrix $\boldsymbol{G}$ of the form:

$$\boldsymbol{G} = \begin{pmatrix} G^{\text{diag}} & G^{\text{off}} \\ G^{\text{off}} & G^{\text{diag}} \end{pmatrix} \tag{13}$$

with $G^{\text{diag}} > G^{\text{off}}$. Let us also suppose the target volumes to have the same magnitude:

$$\boldsymbol{Q} = (Q^1, Q^2) = (Q, \pm Q) \tag{14}$$

After defining $\psi^{(i)}(t) = q^i(t)/Q^i$, the cost becomes:

$$\mathcal{C} = \frac{|Q|^2}{4}\Big[\underbrace{(G^{\text{diag}} + G^{\text{off}})}_{G^{\text{abs}}} \|\psi^{(1)} \pm \psi^{(2)}\|^2 + \underbrace{(G^{\text{diag}} - G^{\text{off}})}_{G^{\text{rel}}} \|\psi^{(1)} \mp \psi^{(2)}\|^2\Big] \tag{15}$$

The interpretation of this result is the following:

- The cost of trading is proportional to the eigenvalues G^{abs} and G^{rel} (where abs and rel stand for absolute and relative mode). It is obviously minimised by choosing $\psi^{(1)} = \psi^{(2)} = \psi^{\star}$, in which case the cost is $|Q|^2 G^{\text{abs/rel}} \|\psi^{\star}\|^2$.
- When trading directionally (ie, if $Q^1 = Q^2$), the minimum cost is proportional to G^{abs}, while the neutral strategy $Q^1 = -Q^2$ yields a smaller cost proportional to G^{rel}.
- One could be tempted to locally trade the cheaper relative mode, but this would construct a long-short position that would have to be closed at a cost later. It is easy to check that the convexity of the cost and the terminal requirement prevent this from being optimal.

Which trajectory is cheaper in terms of risk? If we assume the correlation matrix is given by $\boldsymbol{\rho} = \left(\begin{smallmatrix} 1 & \rho \\ \rho & 1 \end{smallmatrix}\right)$, its eigenvalues are equal to $\Lambda^{\text{abs/rel}} = 1 \pm \rho$. The cost of trading per unit of risk can be written as:

$$\frac{\mathcal{C}}{\mathcal{R}} = \|\psi^{\star}\|^2 \frac{|Q| G^{\text{abs/rel}}}{\sqrt{2\Lambda^{\text{abs/rel}}}} = \|\psi^{\star}\|^2 \frac{|Q| g^{\text{abs/rel}} \sqrt{\Lambda^{\text{abs/rel}}}}{\sqrt{2}} \tag{16}$$

where $(g^{\text{abs/rel}})^{-1}$ is the liquidity in dollars of the absolute and relative modes, respectively. We can interpret this as follows:

- The cost of trading per unit risk depends trivially on $G^{\text{abs/rel}}$, but it has an implicit dependence on the correlation through $\Lambda^{\text{abs/rel}}$. The more correlated the stocks, the more expensive it is to obtain a target risk.
- The liquidity per mode $(g^{\text{abs/rel}})^{-1}$ accounts for both effects, describing how expensive it is to obtain a given target risk by trading either the symmetric or the antisymmetric mode.

B. Fitting the EigenLiquidity model

We present a step-by-step procedure for calibrating our cost model to real data.

(1) Compute the covariance matrix of prices:

$$\boldsymbol{\Sigma} = \mathbb{E}[(\boldsymbol{p}_T - \boldsymbol{p}_0)(\boldsymbol{p}_T - \boldsymbol{p}_0)^{\mathrm{T}}] \tag{17}$$

and extract the volatilities σ^i.

(2) Standardise prices, their covariances and market volumes:

$$x_t^i = p_t^i / \sigma^i \tag{18}$$

$$\rho^{ij} = \Sigma^{ij} / (\sigma_i \sigma_j) \tag{19}$$

$$q_t^i = \sigma^i v_t^i \tag{20}$$

(3) Compute the covariation of prices and volumes:

$$\boldsymbol{r}(t - t') = \mathbb{E}[\dot{\boldsymbol{x}}_t \boldsymbol{q}_{t'}^{\mathrm{T}}] \tag{21}$$

$$\boldsymbol{c}(t - t') = \mathbb{E}[\boldsymbol{q}_t \boldsymbol{q}_{t'}^{\mathrm{T}}] \tag{22}$$

where $\dot{\boldsymbol{x}}_t = (\boldsymbol{x}_{t+\mathrm{d}t} - \boldsymbol{x}_t)/\mathrm{d}t$.

(4) Compute the derivative of the kernel:

$$\dot{\phi}(\tau) = [\phi(\tau + \mathrm{d}\tau) - \phi(\tau)]/\mathrm{d}\tau$$

by solving:

$$\bar{r}(t - t') \propto \int_{-\infty}^{t} \mathrm{d}t'' \dot{\phi}(t - t'') \bar{c}(t'' - t') \tag{23}$$

where:

$$\bar{r}(\tau) = N^{-1} \sum_{i=1}^{N} r^{ii}(\tau)$$

and:

$$\bar{c}(\tau) = N^{-1} \sum_{i=1}^{N} c^{ii}(\tau)/c^{ii}(0).$$

Get the norm from the condition $\phi(0) = 1$.

(5) Extract independent portfolios $\{\boldsymbol{\pi}^a\}_{a=1}^{N}$ from the correlation matrix $\boldsymbol{\rho}$ by computing eigenvectors $\boldsymbol{O}$ and eigenvalues $\boldsymbol{\Lambda}$ according to (7).

(6) Project the covariances $\boldsymbol{r}(\tau)$ and $\boldsymbol{c}(\tau)$ to independent portfolios:

$$\tilde{r}^a(\tau) = (\boldsymbol{\pi}^a)^{\mathrm{T}} \boldsymbol{r}(\tau) \boldsymbol{\pi}^a \tag{24}$$

$$\tilde{c}^a(\tau) = (\boldsymbol{\pi}^a)^{\mathrm{T}} \boldsymbol{c}(\tau) \boldsymbol{\pi}^a \tag{25}$$

(7) Estimate g^a with the maximum-likelihood estimator:

$$g^a = \frac{1}{\Lambda^a}\left(\frac{\int_0^{\infty} \mathrm{d}\tau\, \dot{\phi}(\tau) \tilde{r}^a(\tau)}{\iint_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{d}t'\, \dot{\phi}(t) \dot{\phi}(t') \tilde{c}^a(t - t')}\right) \tag{26}$$

of a stock picked at random in our pool, the other stocks in the pool rise on average by $\bar{G} \approx 0.4 \times 10^{-4}$ times their daily volatility, where $\bar{G}$ is the average of the off-diagonal elements of $\boldsymbol{G}$. Smaller risk modes may be up to 30 times less liquid. Empirically, the liquidity per mode g^a is well fitted by $g^a \propto (\Lambda^a)^{-1/2}$ (see figure 3). This finding is consistent with the assumption of fragmentation invariance, which implicitly requires the parameters $g^a \Lambda^a \to 0$ when $\Lambda^a \to 0$ (see box B).

To illustrate the relevance of these findings when executing a portfolio of trades, let us study numerically a toy daily execution problem of a trader who has target volumes $\boldsymbol{Q}$ corresponding to a fraction $\varphi = 1\%$, 5% or 10% of the daily liquidity of each of $N = 150$ US stocks. We assume the trader uses the optimal synchronous policy derived above: $\boldsymbol{\psi}(t) = \boldsymbol{\psi}^{\star}(t)$.

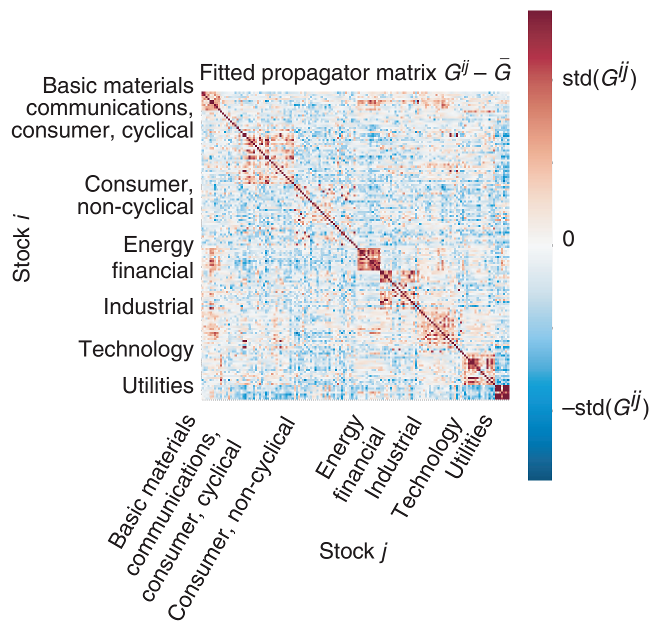
To explore a variety of trading styles, we set the sign ($\epsilon = +$ for buy, $\epsilon = -$ for sell) of the N orders from N biased coin tosses. We vary the bias parameter $\beta = \mathbb{E}[\epsilon^i]$ in the interval $[-1, +1]$. The interpretation of this construction for $\beta = 0$ or ± 1 is very simple:

- for $\beta = \pm 1$, the order is a long or short directional one, and it is strongly exposed to the market mode of risk;
- for $\beta = 0$, the strategy is neutral, and its exposure to the market mode is therefore limited.

The average cost under such an execution policy can be expressed analytically, allowing us to obtain a relation between the cost $\mathcal{C}$ and β:

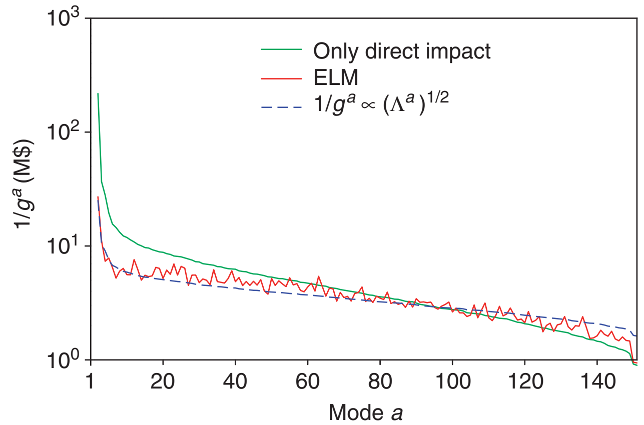
$$\mathbb{E}[\mathcal{C}] = \frac{\varphi^2 \|\psi^{\star}\|^2}{2}\left[(1 - \beta^2) \sum_i G^{ii} (Q_{\mathrm{M}}^i)^2 + \beta^2 \boldsymbol{Q}_{\mathrm{M}}^{\mathrm{T}} \boldsymbol{G} \boldsymbol{Q}_{\mathrm{M}}\right] \tag{27}$$

2 Propagator $\boldsymbol{G}$ fitted during the year 2012 on a sample of 150 US stocks sorted by industrial sectors



The market mode (ie, the average across the entries $\bar{G}$) has been removed in order to highlight the sectorial structure of the market. The figure clearly shows that after hiding the market mode, which accounts for the overall positive interaction between buy (sell) trades and positive (negative) price changes, the other large modes of the propagator matrix can be interpreted as financial sectors, which are responsible for the block structure of the matrix $\boldsymbol{G}$.

3 Liquidity per mode $(g^a)^{-1}$, obtained by normalising the cost of trading the portfolio $\boldsymbol{\pi}^a$ by its corresponding risk (red line)



Each eigenvalue g^a is interpreted as the cost of trading a dollar of risk in the portfolio $\boldsymbol{\pi}^a$, so its inverse gives the liquidity in dollars available on the mode a. The green line represents the liquidity available $\boldsymbol{\pi}^a$ under a model in which no cross-interaction is taken into account, indicating that when neglecting cross-impact one underestimates the cost of trading high-risk modes (eg, the market) and overestimates the cost of trading small-risk ones. The dashed line plotted for comparison indicates the prediction of a model in which $g^a \propto (\Lambda^a)^{-1/2}$. While the liquidity of the large-risk modes (left-hand side of the plot) is relatively easy to estimate from empirical data, a preliminary cleaning step on the noisy part of the eigenvalue spectrum is further required in order to remove the spectrum of the bulk of lower-risk modes on the right-hand side of the plot (Bun *et al* 2016).

where $\boldsymbol{Q}_{\mathrm{M}}$ denotes average dollar risk exchanged by the market on the different stocks.

Comparing (27) with the special cases above, one can see the dollar cost is higher for $\beta = \pm 1$ than for $\beta = 0$ by a factor that can be roughly estimated on the basis of (27) to be

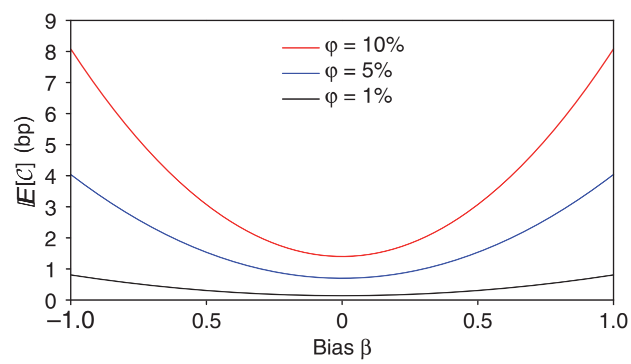
$$g^1 \Lambda^1 / \Big(N^{-1} \sum_a g^a \Lambda^a\Big) \approx 6.6$$

This makes sense, as this is the ratio of the top eigenvalue of $\boldsymbol{G}$, which expresses the cost of trading the market directionally, versus the average of all eigenvalues selecting the direct impact contribution in the left term of (27). Figure 4 shows the full evolution of this ratio as a function of the bias β.

This result holds for a fixed dollar volume traded, but the risk of the resulting position is in fact much higher for $\beta = 1$ than for $\beta = 0$. The cost of trading per unit risk taken, expressed by $\mathbb{E}[\mathcal{C}]/\sqrt{\mathbb{E}[\mathcal{R}^2]}$, is found to be approximately independent of β. By generalising (27) to the ratio $\mathbb{E}[\mathcal{C}]/\sqrt{\mathbb{E}[\mathcal{R}^2]}$, one can relate this finding to the numerical result:

$$g^1 (\Lambda^1)^{1/2} \approx 1.1 \times \sum_a g^a \Lambda^a \Bigg/ \sqrt{\sum_a \Lambda^a}$$

4 Average trading cost $\mathcal{C}$ in basis points for the trading strategy described in the application, as a function of the bias parameter β, for different participation rates of $\varphi = 1\%$, 5% and 10%



Consistent with (27), one can clearly see that directional trading strategies are more expensive in terms of notional traded.

Conclusions

In this paper, we have shown how to leverage the recent quantitative results of Benzaquen *et al* (2017) on cross-impact effects in order to estimate the execution cost of a basket of correlated instruments. We confirm empirically on a pool of 150 US stocks that cross-impact is a very substantial part of the impact of trades on prices. We show that neglecting cross-interactions leads to a distorted vision of the liquidity available on the market: it overestimates the liquidity on high-risk modes and underestimates the liquidity of low-risk modes.

In order to distil these findings into a cost formula, we have assumed the impact matrix has the same eigenvectors as the correlation matrix itself, and the impact eigenvalues are proportional to the risk of the corresponding modes. This specification prevents arbitrage opportunities and price manipulation strategies. It also abides by the principle of fragmentation invariance, which states that trading zero-risk portfolios should have no effect whatsoever on trading costs. We have provided the solution of the corresponding optimal trading problem, which leads to a synchronous U-shaped trading profile across products. This avoids round trips on unwanted positions at a potentially large cost.

In order to keep our approach as simple as possible, we have neglected other sources of cost (spread costs, fees) and considered no risk-aversion effects nor intraday predictive signals. Moreover, we have deliberately disregarded the nonlinear nature of the price impact function, which is known to be better represented by a square-root law (Grinold & Kahn 2000; Schneider & Lillo 2016; Tóth *et al* 2011). These features could be progressively reintroduced into our framework by preserving the idea of an interaction that is diagonal in the space of correlation modes. Indeed, we believe our simpler approach is better suited to transparently illustrating the main effects of cross-impact between financial instruments. ■

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REFERENCES

Alfonsi A and A Schied, 2010
Optimal trade execution and absence of price manipulations in limit order book models
SIAM Journal on Financial Mathematics 1(1), pages 490–522

Alfonsi A and A Schied, 2013
Capacitary measures for completely monotone kernels via singular control
SIAM Journal on Control and Optimization 51(2), pages 1758–1780

Alfonsi A, A Schied and A Slynko, 2012
Order book resilience, price manipulation, and the positive portfolio problem
SIAM Journal on Financial Mathematics 3(1), pages 511–533

Alfonsi A, F Klöck and A Schied, 2016
Multivariate transient price impact and matrix-valued positive definite functions
Mathematics of Operations Research 41(3), pages 914–934

Almgren R and N Chriss, 2001
Optimal execution of portfolio transactions
Journal of Risk 3(2), pages 5–39

Benzaquen M, I Mastromatteo, Z Eisler and J-P Bouchaud, 2017
Dissecting cross-impact on stock markets: an empirical analysis
Journal of Statistical Mechanics: Theory and Experiment 2017, article 023406

Bouchaud J-P, Y Gefen, M Potters and M Wyart, 2004
Fluctuations and response in financial markets: the subtle nature of 'random' price changes
Quantitative Finance 4(2), pages 176–190

Bun J, J Bouchaud and M Potters, 2016
Cleaning correlation matrices
Risk April, pages 54–58

Busseti E and F Lillo, 2012
Calibration of optimal execution of financial transactions in the presence of transient market impact
Journal of Statistical Mechanics: Theory and Experiment 2012(9), article P09010

Curato G, J Gatheral and F Lillo, 2016
Optimal execution with non-linear transient market impact
Quantitative Finance 17(1), pages 41–54

Gatheral J, 2010
No-dynamic-arbitrage and market impact
Quantitative Finance 10(7), pages 749–759

Gatheral J and A Schied, 2013
Dynamical models of market impact and algorithms for order execution
In *Handbook on Systemic Risk*, ed. J-P Fouque and JA Langsam, pages 579–599
Cambridge University Press

Gatheral J, A Schied and A Slynko, 2012
Transient linear price impact and fredholm integral equations
Mathematical Finance 22(3), pages 445–474

Grinold RC and RN Kahn, 2000
Active Portfolio Management
McGraw-Hill, New York

Kratz P and T Schöneborn, 2014
Optimal liquidation in dark pools
Quantitative Finance 14(9), pages 1519–1539

Obizhaeva AA and J Wang, 2013
Optimal trading strategy and supply/demand dynamics
Journal of Financial Markets 16(1), pages 1–32

Schied A, T Schöneborn and M Tehranchi
Optimal basket liquidation for cara investors is deterministic
Applied Mathematical Finance 17(6), pages 471–489

Schneider M and F Lillo
Cross-impact and no-dynamic-arbitrage
Working Paper, arXiv:1612.07742

Schöneborn T, 2016
Adaptive basket liquidation
Finance and Stochastics 20(2), pages 455–493

Tóth B, Y Lempérière, C Deremble, J De Lataillade, J Kockelkoren and J-P Bouchaud, 2011
Anomalous price impact and the critical nature of liquidity in financial markets
Physical Review X 1(2), article 021006

Wang S, 2017
Trading strategies for stock pairs regarding to the cross-impact cost
Working Paper, arXiv:1701.03098

Wang S and T Guhr, 2016
Microscopic understanding of cross-responses between stocks: a two-component price impact model
SSRN Working Paper, available at *https://ssrn.com/abstract=2892266*

Wang S, R Schäfer and T Guhr, 2015
Price response in correlated financial markets: empirical results
Working Paper, arXiv:1510.03205