Tutorat de Mathématiques

Tutorat 1: Equation des films minces

On s'intéresse aux propriétés d'une équation aux dérivées partielles décrivant la relaxation des films minces purement visqueux.

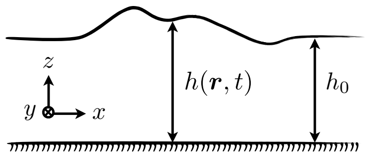

Figure 1: Schéma

Dans le cadre de l'approximation de lubrification, l'évolution spatiotemporelle de l'épaisseur $h(\boldsymbol{r}, t)$ d'un film liquide mince posé sur un substrat plan (voir Fig. 1) est régie par l'équation ci-dessous:

$$\partial_t h + \frac{\gamma}{3\eta} \boldsymbol{\nabla} \cdot \left(h^3 \boldsymbol{\nabla} \Delta h\right) \quad = \quad 0 \;, \tag{1}$$

où γ désigne la tension de surface liquide-air, η la viscosité dynamique du liquide et $\Delta = \boldsymbol{\nabla}^2$ l'opérateur laplacien scalaire. Afin d'obtenir cette équation, les effets liés à la gravité ainsi que les effets inertiels ont été négligés, les conditions aux limites ont été choisies de non-glissement au substrat et de non-contrainte à la surface libre, et enfin les pentes de la surface ont été supposées faibles à tout instant.

I Recherche bibliographique

1. Effectuer une recherche bibliographique précise et bien documentée sur l'équation des films minces (thin film equation). Cela veut dire présenter un petit nombre d'ouvrages, articles et articles de revue judicieusement choisis et résumer leur contenu en quelques mots.

II Equation des films minces

2. A l'aide de la bibliographie, donner une démonstration rigoureuse de l'équation des films minces.

Afin de simplifier les calculs, on choisit de limiter l'étude à une dimension. Le problème est alors supposé invariant suivant la direction y, de sorte que $h(\boldsymbol{r}, t) = h(x, t)$.

3. Montrer que l'équation (1) devient:

$$\partial_t h + \frac{\gamma}{3\eta} \left(h^3 h'''\right)' \quad = \quad 0 \;, \tag{2}$$

où " $'$ " signifie la dérivée spatiale.

4. Adimensionner l'équation (2) et absorber les facteurs numériques. On notera h_0 l'épaisseur du film à l'équilibre.

5. Montrer qu'en linéarisant l'équation obtenue autour de l'équilibre on obtient une équation de la forme:

$$\partial_t \zeta + \zeta'''' \quad = \quad 0 \;, \tag{3}$$

où ζ est le déplacement adimensionné de la surface libre par rapport à l'équilibre. Commenter.

III Conditions aux limites et unicité des solutions

On considère maintenant le problème aux limites que constitue l'équation (3) avec:

$$\zeta(x, t = 0) \quad = \quad f(x) \qquad x \in \mathbb{R} \qquad \text{avec} \int \mathrm{d}x\, f(x) < \infty \tag{4}$$

$$\lim_{x \to \pm\infty} \zeta(x, t) \quad = \quad 0 \qquad \text{pour tout } t \;. \tag{5}$$

6. Comment appelle-t'on ce type de conditions aux limites ?

7. Démontrer l'unicité des solutions de ce problème.

IV Solution générale

8. Discuter brièvement de l'existence de solutions de l'équation (3).

9. Montrer que la fonction de Green de l'opérateur linéaire décrivant l'équation (3) s'écrit:

$$\mathcal{G}(x,t) \quad = \quad \frac{\Theta(t)}{2\pi}\int \mathrm{d}k\, e^{-k^4 t} e^{ikx} \ , \tag{6}$$

où Θ désigne la fonction d'Heaviside.

10. Donner sous la forme d'un produit de convolution la solution au problème posé en partie **III**.

V Solutions auto-similaires

11. Donner quelques examples de problèmes physique admettant des solutions autosimilaires[1].

On cherche maintenant des solutions autosimilaires de l'équation (3) de la forme $\zeta(x,t) = t^\alpha f(u)$ avec $u = xt^\beta$.

12. Déterminer les valeurs de α et β. Commenter.

13. Donner l'équation différentielle ordinaire vérifiée par $f(u)$. La résoudre à l'aide d'un logiciel de calcul formel, puis représenter la solution.

14. En posant le changement de variable $u = xt^\beta$, $q = kt^{-\beta}$, montrer que la fonction de Green peut s'écrire sous la forme $\mathcal{G}(x,t) = t^\beta \Theta(t)\phi(u)$ où $\phi(u)$ est une fonction à determiner.

VI Solution asymptotique intermédiaire et universalité

15. Dans le cas où $\int \mathrm{d}x\, f(x) = \mathcal{M}_0 \neq 0$, montrer que $(t^\beta \mathcal{M}_0)^{-1}\zeta(x,t)$ converge uniformément[2] par rapport à la variable t vers $\phi(u)$.

16. Discuter des termes *solution asymptotique intermédiaire* et *universalité* à la lumière de la réponse à la question précédente.

17. Représenter à l'aide d'un logiciel de calcul formel l'évolution d'une condition initiale $f(x)$ de votre choix, ainsi que la convergence de la solution correctement normalisée vers la solution intermédiaire asymptotique en variable u.

VII Un peu de physique

18. Discuter de l'analogie que existe entre cette étude de l'équation des films minces et l'étude de l'explosion atomique par G. I. Taylor. On pourra à cet effet s'aider de l'introduction de l'ouvrage de Barenblatt fournie en annexe. Citer d'autres examples.

19. Montrer en utilisant l'équation (2) que l'énergie initialement disponible sous forme d'énergie de surface est intégralement dissipée sous forme visqueuse pendant la relaxation. Discuter de la validité du résultat obtenu à la question **15.** pour les solutions l'équation (2).

20. Question bonus Démontrer que pour tout $u \in \mathbb{R}$ on a:

$$\frac{1}{2\pi}\int \mathrm{d}q\, e^{-q^4} e^{iqu} \quad = \quad \frac{1}{\pi}\Gamma\left(\frac{5}{4}\right)\ {}_0H_2\left[\left\{\frac{1}{2},\frac{3}{4}\right\},\frac{u^4}{256}\right] - \frac{u^2}{8\pi}\Gamma\left(\frac{3}{4}\right)\ {}_0H_2\left[\left\{\frac{5}{4},\frac{3}{2}\right\},\frac{u^4}{256}\right] \ , \tag{7}$$

où la fonction hypergeometric de classe $(0,2)$ est définie par:

$${}_0H_2\left(\{a,b\},w\right) \quad = \quad \sum_{k\geq 0} \frac{1}{(a)_k (b)_k}\frac{w^k}{k!} \ , \tag{8}$$

et où $(.)_k$ est la notation dite de Pochhammer signifiant la factorielle croissante.

[1] Autosimilaire = qui conserve sa forme par une transformation simple des variables d'espace et de temps.

[2] On rappelle qu'une fonction de deux variable $g(x,y)$ converge uniformément par rapport à la variable y vers la limite $l(x)$ si et seulement si $\lim_{y\to\infty} ||g(x,y) - l(x)||_{\infty,x} = 0$, où $||...||_{\infty,x} = \sup\{|...|, x \in \mathbb{R}\}$.

similarity, in only slightly modified form. However, the central part of this book is entirely new: in particular I have replaced some complicated and difficult basic examples with simpler ones.

I want to express my thanks to Cambridge University Press (Dr D. Tranah and Dr A. Harvey). In fact, the very idea that I should write such an 'intermediate' book matching my inaugural lecture (Barenblatt 1994) and the large book (Barenblatt 1996) belongs with these gentlemen.

I want to express my gratitude to Professor V.M. Prostokishin, who attended all my lectures and gave me important advice both about the lectures and the present book. I am grateful to Professor L.C. Evans and Professor M. Brenner for reading the manuscript and for valuable comments. I want to thank Professors S. Kamin, R. Dal Passo, M. Bertsch, N. Goldenfeld, D.D. Joseph, L.A. Peletier, G.I. Sivashinsky and J.L. Vazquez for the stimulating and friendly exchange of thoughts concerning the subjects presented in this book over many years. I thank Mrs Deborah Craig for processing the manuscript.

To my friend Alexandre Chorin I want to express special thanks for our remarkable time in Berkeley. I have learned from him a lot, in particular his basic paradigm of computational science: this is a different, independent and very productive way of mathematical modelling. I hope to be able to use this knowledge in my future work.

Introduction

The term *scaling* describes a seemingly very simple situation: the existence of a power-law relationship between certain variables y and x,

$$y = Ax^{\alpha}, \tag{0.1}$$

where A, α are constants. Such relations often appear in the mathematical modelling of various phenomena, not only in physics but also in biology, economics, and engineering. However, scaling laws are not merely some particularly simple cases of more general relations. They are of special and exceptional importance; scaling never appears by accident. Scaling laws always reveal an important property of the phenomenon under consideration: its *self-similarity*. The word 'self-similar' means that a phenomenon reproduces itself on different time and/or space scales – I will explain this later in detail.

I begin with one of the most illuminating examples of the discovery of scaling laws and self-similar phenomena: G.I. Taylor's analysis of the basic intermediate stage of a nuclear explosion. At this stage a very intense shock wave propagates in the atmosphere and the gas motion inside the shock wave can be considered as adiabatic.

This work started in one of the worst and most alarming days of the Battle of Britain, in the early autumn of 1940. Cambridge professor Geoffrey Ingram Taylor was invited to a business lunch at the Athenaeum by Professor George Thomson, chairman of the recently appointed MAUD committee (the name 'MAUD' originally appeared by chance, but later it was interpreted as the acronym for 'military application of uranium detonation'). G.I. Taylor was told that it might be possible to produce a bomb in which a very large amount of energy would be released by nuclear fission – the name 'atomic bomb' had not yet been used. The question was: what mechanical effect might be expected if such an explosion were to occur? The answer would be of crucial importance for the further development of events. Shortly before this conversation the confidential

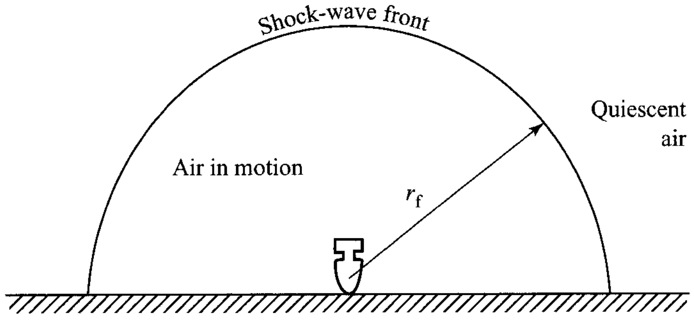

Figure 0.1. A very intense shock wave propagating in quiescent air.

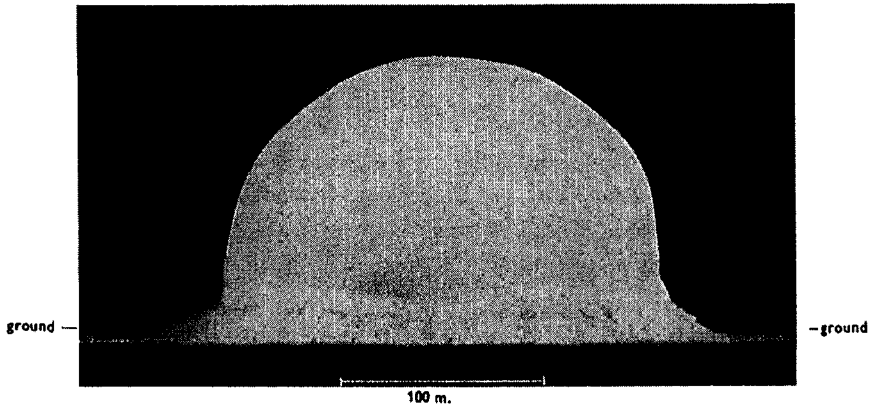

Figure 0.2. Photograph of the fireball of the atomic explosion in New Mexico at $t = 15$ ms, confirming in general the spherical symmetry of the gas motion (Taylor 1950b, 1963).

report of G.B. Kistyakovsky, the well-known American expert in explosives, had been received. Kistyakovsky claimed that even if the bomb were successfully constructed and exploded, its mechanical effect would be much less than expected because the main part of the released energy would be lost to radiation. As R.W. Clark wrote in his instructive book (Clark 1961), in the whole of Britain there was only one man able to solve this problem – Professor G.I. Taylor.

To answer this question, G.I. Taylor had to understand and calculate the motion of the ambient gas after such an explosion. It was clear to him that, after a very short initial period (related as we now know to thermal-wave propagation in quiescent air), a very intense shock wave would appear (Figure 0.1). The motion was assumed to be spherically symmetric, that is, identical for all radii going out from the explosion centre. (This simplifying assumption later received excellent confirmation in the first atomic test; see Figure 0.2.) For constructing

a complete mathematical model the following partial differential equations of motion inside the shock wave had to be considered:

1. the equation for the conservation of mass;
2. the equation for the conservation of momentum;
3. the equation for the conservation of energy.

It was intuitively clear to G.I. Taylor that at this early stage in the explosion viscous effects could be neglected and the gas motion could be considered as adiabatic. The above equations of motion had to be supplemented by the following boundary conditions at the shock-wave front:

1. the condition for the conservation of mass;
2. the condition for the conservation of momentum;
3. the condition for the conservation of energy.

Also, the initial conditions, at the beginning of the very intense shock-wave-propagation stage of a nuclear explosion, had to be prescribed.

In fact, this primary mathematical model is so complicated that even now nobody is able to treat it analytically. Adequate computing facilities at that time were non-existent. Moreover, the problem formulation outlined above is incomplete, because nobody knew then or knows now how the air density, air pressure and air velocity are distributed inside the initial shock wave at the time when the shock wave just outstrips the thermal wave and the adiabatic gas motion begins.

G.I. Taylor, however, was astute. His ability to deal with seemingly unsolvable problems, by apparently minor adjustment converting them to problems admitting simple and effective mathematics, was remarkable. And here also he took several steps, of crucial importance, which allowed him to obtain the solution that was needed in a simple and effective form. In addition his formulation allowed him to overcome the lack of detailed knowledge of the initial distribution of the gas density, pressure and velocity. G.I. Taylor's steps were as follows:

1. He replaced the problem by an 'ideal' one. As he wrote (see Taylor 1941, 1950a, 1963), this ideal problem is the following: '*A finite amount of energy is suddenly released in an infinitely concentrated form.*' This means that r_0, the initial radius of the shock wave (the radius at which the shock wave outstrips the thermal wave), is taken as equal to zero, that is, the explosion is considered as instantaneous and coming from a point source of energy. It is clear that neglecting the initial radius of the shock wave r_0 is allowable (if at all!) only when the motion is considered at a stage when the shock front radius r_f is much larger than r_0. If the initial shock-wave

radius is taken as equal zero then the initial distributions of the air density, pressure and velocity inside the initial shock wave disappear from the problem statement: a great simplification.
2. At the same time, he restricted himself to consideration of the motion at the stage when the maximum pressure of the moving gas, reached at the shock-wave front, is large, much larger than the pressure p_0 in the ambient air; this allowed him to neglect the terms involving the initial pressure p_0 in the boundary conditions at the shock-wave front and in the initial conditions. Note that, namely, this stage determines the mechanical effect of the explosion.

The first question G.I. Taylor addressed was: what are the quantities on which the shock-wave radius r_f depends? In the original 'non-ideal' problem they are obviously:

1. E, the total explosion energy, concentrated in the sphere of radius r_0 where the shock wave outstrips the thermal wave (according to the second assumption above the initial internal energy of the ambient quiescent air is negligible);
2. ρ_0, the initial density of the ambient air;
3. t, the time reckoned from the moment of explosion;
4. r_0, the initial radius of the shock wave;
5. p_0, the pressure of the ambient quiescent air;
6. γ, the adiabatic index.

The units for measuring these quantities in the c.g.s. system of units are

$$[E] = \frac{\mathrm{g\,cm^2}}{\mathrm{s^2}},\ [\rho_0] = \frac{\mathrm{g}}{\mathrm{cm^3}},\ [t] = \mathrm{s},\ [r_0] = \mathrm{cm},\ [p_0] = \frac{\mathrm{g}}{\mathrm{cm\,s^2}}; \tag{0.2}$$

γ is a dimensionless number. We shall see later how important it was that G.I. Taylor neglected the last two quantities r_0 and p_0, thus replacing the problem by an ideal one.

The reader may ask a natural question: in the real explosion r_0 and p_0 are certain positive numbers which definitely influence the whole gas motion from the very beginning to the end. How can their values be taken to be equal to zero?

In fact (and this comment will be important in our future analysis), the real content of Taylor's assumption was that *at the intermediate stage under consideration, where the mechanical effect occurs*, the motion remains the same if we replace r_0 by λr_0, and p_0 by μp_0. Here λ and μ are arbitrary positive numbers 'of order unity'. This will be explained in detail in Chapter 5, but those who are familiar with the idea of a transformation group even vaguely, will recognize that in fact this was an assumption of group invariance at the all-important intermediate stage.

Taylor's next step can be represented in the following way. He introduced the quantity

$$R = \left(\frac{Et^2}{\rho_0}\right)^{1/5}, \tag{0.3}$$

which is measured according to (0.2) in units of length. Then, if we replace centimeters, cm, by another unit of length, m, mm, μm, km, ..., or in general by cm divided by an arbitrary positive number L, the value of R will be magnified by L, as will also the value of r_f, whereas the quantity

$$I = \frac{r_f}{R} \tag{0.4}$$

obviously will remain unchanged.

The quantity I depends in principle on the same quantities as r_f, and this dependence can be represented, neglecting r_0 and p_0, as

$$I = \frac{r_f}{R} = F(R, \rho_0, t, \gamma) \tag{0.5}$$

where F is a certain function which is not known. The arguments r_0 and p_0 were neglected by Taylor: this was, as we will see, a step of crucial importance. The argument γ is an numerical constant.

The first three arguments of F have independent dimensions. This means, in particular, that time t is measured in time units, i.e., seconds or otherwise s/T where T is an arbitrary positive number. Units of time are absent in the dimensions of the first two arguments; therefore, by varying the number T we can vary the numerical value of the argument t while leaving the values of I and those two other arguments of I invariant (all three others, in fact, since γ is a fixed number). But this means exactly that I cannot depend on t. Similarly with ρ_0: if we vary the unit of mass then the value of ρ_0 will vary arbitrarily, leaving I and the first argument R invariant. That means that I likewise does not depend on ρ_0. Furthermore, I does not depend on the argument R: by varying the unit of length we vary R, but the value of I remains invariant. Thus, the function F is simply a constant depending on the value of γ, and so Taylor's famous scaling law for the radius of the shock wave was obtained:

$$r_f = C(\gamma)\left(\frac{Et^2}{\rho_0}\right)^{1/5}, \tag{0.6}$$

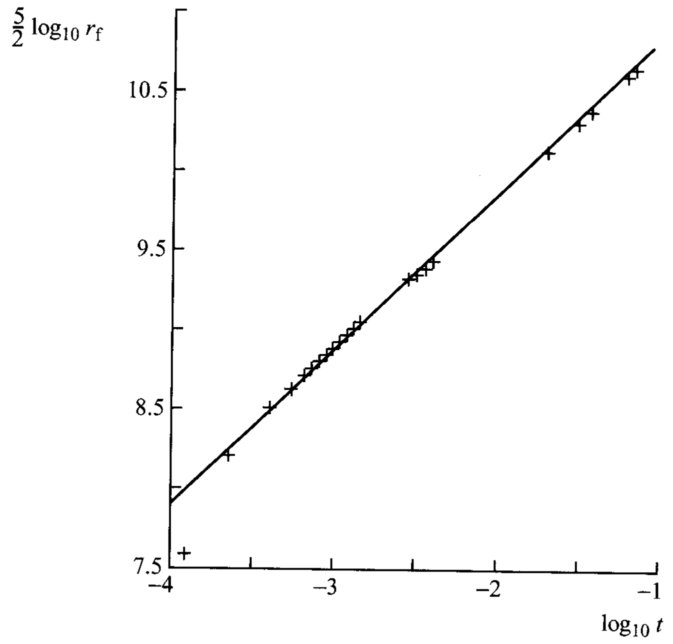

Figure 0.3. Logarithmic plot of the fireball radius, showing that $r_f^{5/2}$ is proportional to the time t (Taylor 1950b, 1963).

or, in the logarithmic form that he used,

$$\frac{5}{2}\log_{10} r_f = \frac{5}{2}\log_{10} C + \frac{1}{2}\log_{10}\left(\frac{E}{\rho_0}\right) + \log_{10} t. \qquad (0.7)$$

Later, Taylor's processing of the photographs taken by J.E. Mack of the first atomic explosion in New Mexico in July 1945 (Taylor 1950b, 1963) confirmed this scaling law (Figures 0.2 and 0.3) – a well-deserved triumph of Taylor's intuition. We can see how important it was to neglect the arguments r_0 and p_0, the initial radius of the shock wave and the initial pressure. If not, additional variable arguments would have appeared in the function F and we would have returned to the hopeless mathematical model that we faced at the outset. But the outcome for the simplified situation was different. Taylor was able to obtain in the same way scaling laws for the pressure, velocity and density immediately behind the shock-wave front:

$$p_f = C_p(\gamma)\left(\frac{E^2\rho_0^3}{t^6}\right)^{1/5}, \qquad \rho_f = C_\rho(\gamma)\rho_0, \qquad u_f = C_u(\gamma)\left(\frac{E}{t^3\rho_0}\right)^{1/5}. \qquad (0.8)$$

Inside the shock wave an additional argument, the distance r from the center of the explosion, appears, so that the relationships for the pressure, density and velocity inside the shock wave can be represented in the form

$$p = p_f P\left(\frac{r}{r_f}, \gamma\right), \qquad \rho = \rho_f R\left(\frac{r}{r_f}, \gamma\right), \qquad u = u_f V\left(\frac{r}{r_f}, \gamma\right). \qquad (0.9)$$

The structure of the relationships (0.9) obtained by Taylor is instructive. It demonstrates that the phenomenon has the important property of *self-similarity*. This means that the spatial distribution of pressure (and other quantities) varies with time while remaining always geometrically similar to itself (Figure 0.4(*a*)): the distribution at any time can be obtained from that at a different time by a simple similarity transformation. Therefore in 'reduced' coordinates using p_f, ρ_f, u_f and r_f as corresponding scales,

$$\frac{p}{p_f}, \qquad \frac{\rho}{\rho_f}, \qquad \frac{u}{u_f}, \qquad \text{and } \frac{r}{r_f},$$

the spatial distributions of pressure, density and velocity remain invariant in time (Figure 0.4(*b*)). The property of self-similarity greatly simplifies the investigation: instead of the two independent variables r and t in the system of differential equations, boundary conditions and initial conditions mentioned above, Taylor obtained one single variable argument, r/r_f, in his solution and so was able to reduce the original problem, which required the solution of partial differential equations to the solution of a set of ordinary differential equations. The method of solution was sufficiently simple that he himself was able to make all the necessary numerical computations using a primitive calculator. In particular, he showed that the constant C in the scaling law (0.6) is close to unity: for $\gamma = 1.4$, $C = 1.033$.

G.I. Taylor submitted his paper on Friday 27 June 1941. The great American mathematician J. von Neumann, who was also involved in the atomic problem and asked the same question independently, submitted a paper three days later, on Monday 30 June 1941 (von Neumann 1941; see also von Neumann 1963). His solution complemented Taylor's solution – he noticed an energy integral for the set of ordinary differential equations and was able to obtain the solution in closed form. Later, the solution of this problem was published in the Soviet Union by L.I. Sedov (Sedov 1946, 1959), who also found the energy integral, and by other authors, R. Latter (1955) and J. Lockwood Taylor (1955).

We have seen that in obtaining the scaling law (0.6) and achieving the property of self-similarity an important role was played by dimensional analysis: the construction of dimensionless quantities from the arguments of the function

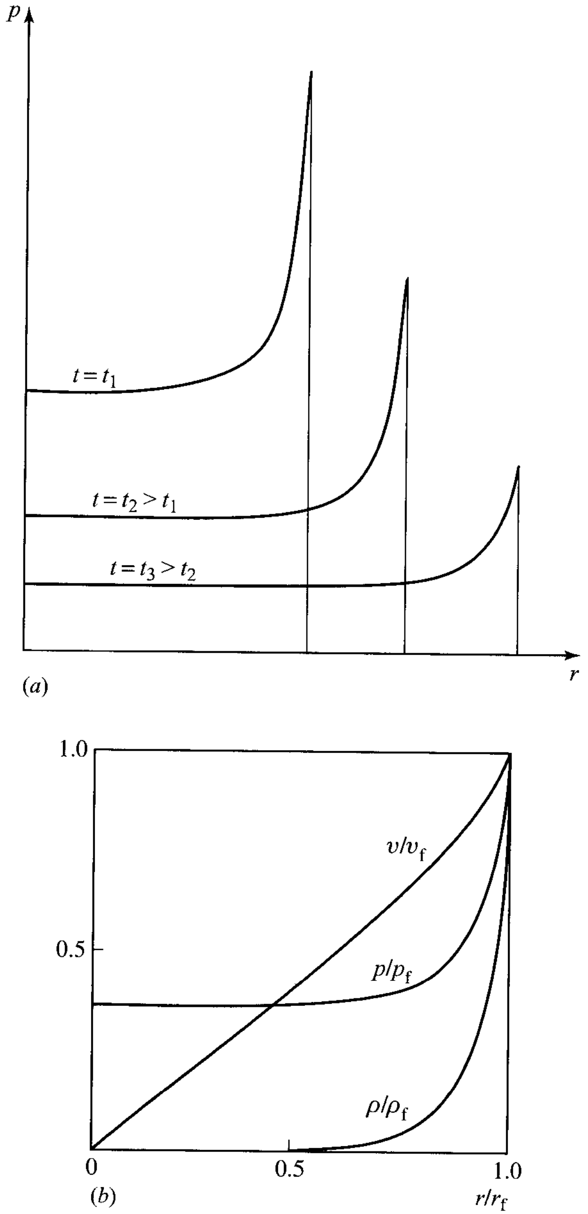

Figure 0.4. (*a*) Air pressure as a function of radius at various instants of time for the motion of air following an atomic explosion. The pressure distributions at various times are similar to one another. (*b*) Spatial distributions of the gas pressure, density and velocity in the reduced 'self-similar' coordinates ρ/ρ_f, p/p_f, u/u_f and r/r_f do not depend on time.

F with subsequent reduction in the number of arguments. The idea on which dimensional analysis is based is fundamental, but very simple: physical laws cannot depend on an arbitrary choice of basic units of measurement. The formal recipe for using dimensional analysis is very simple also. The main art, however, is not in using this simple tool but in finding, as G.I. Taylor did, the proper formulation or idealization of the problem in hand – an instantaneous concentrated very intense explosion in his case – that allows effective use of this tool. Here the key point is the concept of *intermediate asymptotics*: consideration of the phenomenon in intermediate time and space intervals.

It is important, however, to note that dimensional analysis is not always sufficient for obtaining self-similar solutions and scaling laws. Moreover, it can be claimed that as a rule it is not so and that the Taylor–von Neumann solution to the explosion problem was in fact a rare and lucky exception.

Here an instructive role is played by the paper by K.G. Guderley (1942) where, in a certain sense, the mirror image of the problem of a very intense explosion was considered. The formulation of this implosion problem is as follows.[1] On the wall of a spherical cavity of radius r_0 in an absolutely rigid vessel filled by gas of density ρ_0 (Figure 0.5) there is a uniform thin layer of a strong explosive. The latter is exploded instantaneously and uniformly over the wall and a strong spherical shock wave is formed. The shock wave converges to the center of the cavity. It is very intense, as in the case of a very intense explosion, so that the pressure behind the wave is much larger than the initial gas pressure p_0, which, as in the case of a very intense explosion, can be neglected. The shock wave comes to a focus at the center of the cavity at a time which we take as $t = 0$, so that the time before focusing will be negative, $t < 0$. Similarly to the case of an intense explosion, dimensional analysis gives for the radius of the shock wave

$$r_f = [E(-t)^2/\rho_0]^{1/5}\Phi(\eta, \gamma), \qquad \eta = \frac{r_0}{[E(-t)^2/\rho_0]^{1/5}} \tag{0.10}$$

where as before E is the energy of the explosion and γ is the adiabatic index.

Seemingly the application of reasoning analogous to that for the case of an intense explosion would suggest that the argument η goes to infinity at $t \to 0$ and therefore can be neglected close to the focus, so that a formula analogous to (0.6) could be obtained:

$$r_f = C(\gamma)\left[\frac{E(-t)^2}{\rho_0}\right]^{1/5} \tag{0.11}$$

[1] A detailed discussion of the Guderley problem can also be found in the books by Zeldovich and Raizer (1967) and Landau and Lifshitz (1987).

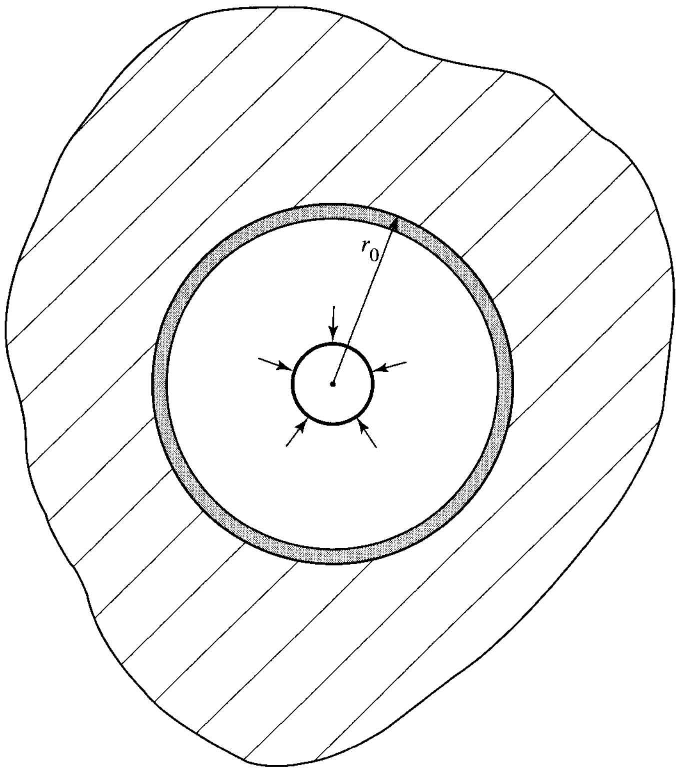

Figure 0.5. A very intense implosion in a spherical cavity. The explosive is placed on the wall of the cavity. The black dot shows the shock front as it comes to a focus at the centre of the cavity at $t = 0$.

In fact, this is not the case, for the following reason. In the case of implosion the function $\Phi(\eta, \gamma)$ at $\eta \to \infty$ does not tend to a finite non-zero limit as was the case for an explosion! However, it happens that at $\eta \to \infty$ the function $\Phi(\eta, \gamma)$ has a power-law-type behavior, $\Phi(\eta, \gamma) \sim C(\gamma)\eta^{-\beta}$ where $\beta = \beta(\gamma) =$ const > 0, so that at $t \to 0$, that is, close to the focus, the expression for the radius of the shock wave assumes the form

$$r_f = C(\gamma) r_0^{-\beta} \left[\frac{E(-t)^2}{\rho_0} \right]^{\alpha/2} = A(-t)^{\alpha},$$

$$\alpha = \frac{2}{5}(1+\beta), \qquad A = C(\gamma) r_0^{-\beta} \left(\frac{E}{\rho_0} \right)^{\alpha/2}. \tag{0.12}$$

It is important to note that the exponent α cannot be obtained by dimensional analysis, as it was in the case of an intense explosion, but requires a more complicated technique, the solution of a *nonlinear eigenvalue problem.*

The Guderley (1942) solution as well as the solution to the 'impulsive load' problem which is in fact a one-dimensional analog of the implosion problem, obtained by von Weizsächer (1954) and Zeldovich (1956), introduced a new class of self-similar phenomena: *incomplete similarity and self-similar solutions of the second kind.* These problems are closely related to the concept of the renormalization group, well known in theoretical physics.

In what follows we will present in detail the ideas of dimensional analysis, physical similarity, self-similarity, intermediate asymptotics and the renormalization group. Our goal is to demonstrate in detail the many possibilities for application of these ideas and also the difficulties which can occur – throughout using many examples. Most of the examples in the present book are related to fluid dynamics: my experience shows that the elements of fluid mechanics are familiar to engineers, mathematicians and physicists. Those who are more interested in elasticity, fracture, fatigue or geophysical fluid dynamics can find additional examples in my book Barenblatt (1996). The examples ('Problems') considered in the present book should be considered as an essential part of the whole text.