High Reynolds statistical modelling of wave-vortex interactions, from gravity waves to acoustic ones

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Outline

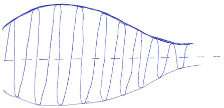
- A General context in **3D turbulence in fluids** with waves. Non-propagating modes coexisting with dispersive wave modes.
- General strategy using a QNM (Quasi-Normal Markovian) ingredient for 'weak' and 'strong' turbulence.
- The toroidal cascade vs. gravity waves turbulence in stably-stratified turbulence
- Role of N/f in rotating stably-stratified turbulence
- Weakly compressible homogeneous isotropic turbulence. Solenoidal mode vs. acoustic waves and pseudo-sound
- Conclusions and perspectives about the use of multimodal, possibly anisotropic, holistic triadic spectral closure, EDQNM and beyond

Basic formalism before statistical approach

Identifying non-propagating modes and wave-modes

- Linear basic eigenmodes decomposition prior to Wave turbulence theory $\hat{\boldsymbol{v}} = a_0(\boldsymbol{k})\boldsymbol{N}^{(0)} + a_1(\boldsymbol{k})\boldsymbol{N}^{(1)}e^{+\imath\sigma_k t} + a_{-1}(\boldsymbol{k})\boldsymbol{N}^{(-1)}e^{-\imath\sigma_k t}$, with σ_k the dispersion law (continuous 3D wave-space)
- Replace the constants by time-dependent amplitudes $a_0(\boldsymbol{k}, t)$, $a_{\pm 1}(\boldsymbol{k}, t)$ to be substituted to $\hat{\boldsymbol{u}}$ variables

Slow amplitudes $a_s(\boldsymbol{k}, \epsilon t)$



vs. *rapid* phases $e^{\pm\imath\sigma_k t}$

Caveat on the nature of the non-propagating, 'vortex', mode

- Two kinds of modes with zero wave-frequency, a fully 3D one, a_0 and the zero-limit of the wave modes, $(a_\pm, \sigma_k = 0)$, with lower dimension

$$\boldsymbol{v} = \left(a_0 + a_+ e^{+\imath\sigma_k t} + a_- e^{-\imath\sigma_k t}\right) e^{\imath \boldsymbol{k}\cdot\boldsymbol{x}}$$

- The so-called *wave-vortex* decomposition is often not intrinsic to the physics of fluids, with examples
  -) 2D mode in purely rotating turbulence? $\sigma_k = 2\Omega\frac{k_\parallel}{k}$, integrable singularity ($k_\parallel = 0$, with $a_0 = 0$), in the unbounded case; no longer in the *bounded case* (Scott, JFM, 2014).
  -) The quasi-geostrophic mode in stably-stratified turbulence with (and without) rotation? Unbounded case, f-plane approx., a_0 is a 3D toroidal mode, $k_\perp = 0$, $\sigma_k = N\frac{k_\perp}{k}$, is the VSHF (1D) mode (without rotation); it is a propagating mode (Rossby waves!) in *the β-plane approx.*

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Flow configuration	Propagating / non-propagating modes: Non-propagating mode(s)	Wave mode(s)	Dispersion law
Purely rotating	None	$\underbrace{u^{(2)} \pm \imath u^{(1)}}_{\text{inertial w.}}$	$\sigma_r = 2\Omega \frac{k_\parallel}{k}$
Purely stratified	$\underbrace{u^{(1)}}_{\text{toroidal}}$	$\underbrace{u^{(3)} \pm \imath u^{(2)}}_{\text{gravity w.}}$	$\sigma_s = N \frac{k_\perp}{k}$
Rotating & stratified	$\underbrace{\frac{\sigma_s}{\sigma_{rs}} u^{(1)} + \frac{\sigma_r}{\sigma_{rs}} u^{(3)}}_{\text{Quasi-Geostrophic}}$	$\underbrace{-\frac{\sigma_r}{\sigma_{sr}} u^{(1)} + \frac{\sigma_s}{\sigma_{sr}} u^{(3)} + \imath u^{(2)}}_{\text{inertia-gravity w.}}$	$\sigma_{rs} = \sqrt{\sigma_r^2 + \sigma_s^2}$
Weakly compressible	$\underbrace{u^{(1)}, u^{(2)}}_{\text{solenoidal}}$	$\underbrace{u^{(4)} \pm \imath u^{(3)}}_{\text{acoustic w.}}$	$\sigma_c = c_0 k$

Table 0.1. *Linear eigenmodes, Vortex-Wave-type and dispersion law for different flow cases*

- Linear combination from a toroidal - poloidal - dilatational decomposition of velocity (see Sagaut & CC, Springer 2018, and 'bibles' on geophysics, e.g. Pedlovsky)

$$\hat{\boldsymbol{u}} = \underbrace{u^{(1)}\boldsymbol{e}^{(1)}}_{\text{toroidal}} + \underbrace{u^{(2)}\boldsymbol{e}^{(2)}}_{\text{poloidal}} + \underbrace{u^{(3)}\boldsymbol{e}^{(3)}}_{\text{dilatational}}$$

- Anisotropic dispersion laws, possibly a low-dimension zero wave-mode if $\sigma(\boldsymbol{k}) = 0$.

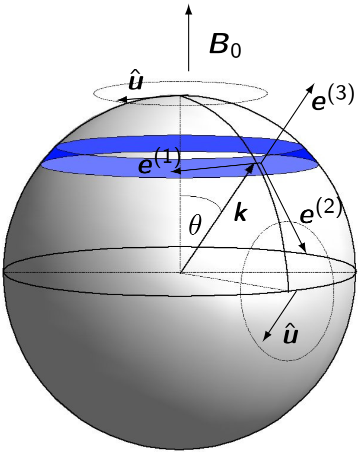


Figure: Craya-Herring frame $(e^{(1)}, e^{(2)}, e^{(3)})$ in Fourier space.

Exact equations for 'slow' amplitudes

Phase of the $\boldsymbol{k}$-mode: $\exp\left(\imath(\boldsymbol{k}\cdot\boldsymbol{x} + s_k\sigma_k t)\right)$, $s_k = 0, \pm 1$

Inject $\hat{\boldsymbol{v}} = \sum_{s=0,\pm 1} a_s(\boldsymbol{k}, t)\boldsymbol{N}^s e^{\imath s\sigma_k t}$ into Navier-Stokes-Bousinesq-type equations for $\boldsymbol{v}$:

$$\dot{a}_s(\boldsymbol{k}, t) = \sum_{s', s''=0,\pm 1} \int G_{kpq}^{ss's''} e^{\imath(s\sigma_k + s'\sigma_p + s''\sigma_q)t} a_{s'}(\boldsymbol{p}, t) a_{s''}(\boldsymbol{q}, t) d^3\boldsymbol{p},$$

with $s, s', s'' = 0, \pm 1, \qquad \boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = \boldsymbol{0}.$

Statistical approach for multipoint correlations

General closure strategy

Transferring the machinery of EDQNM from $\hat{\boldsymbol{u}}$ to *slow* amplitudes.

- The typical equation for three-point third-order correlations to be closed:

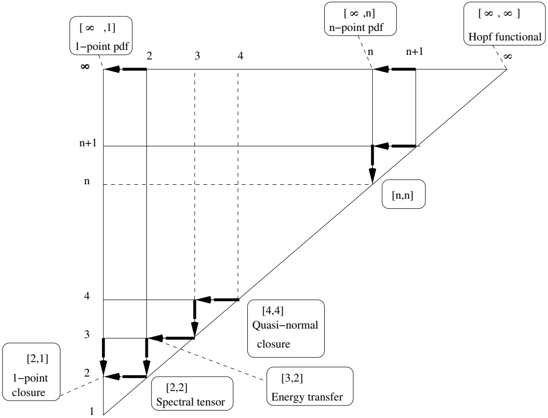
$$\left(\frac{\partial}{\partial t} + \nu(k^2+p^2+q^2) + \imath(s\sigma(\boldsymbol{k}) + s'\sigma(\boldsymbol{p}) + s''\sigma(\boldsymbol{q}))\right) S_{ss's''}(\boldsymbol{k},\boldsymbol{p},t) =$$

$$= T^{(QN)}_{ss's''} + C^{(IV)}_{ss's''}, \quad s, s', s'' = 0, \pm 1, \quad \boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} = \boldsymbol{0}$$

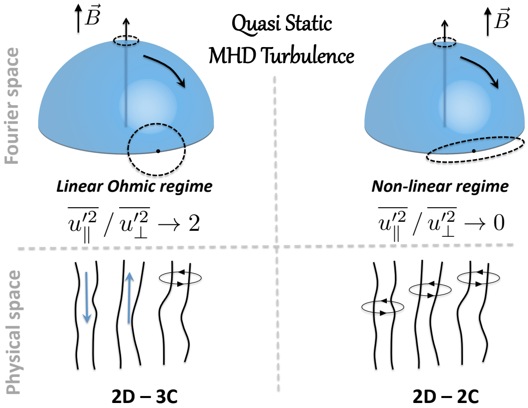
- Classical approach to wave turbulence,
  QN ($C^{IV} = 0, \langle vvvv\rangle = \sum\langle vv\rangle\langle vv\rangle$) (*e.g. Benney and Newell, 1969*)
  $\leftrightarrow$ Random Phase Approximation
  Markovianisation $\leftrightarrow$ two time-scales t and ϵt, final equations in terms of slow variables only.
- Including an additional Eddy Damping ingredient as in EDQNM for HIT, for the zero mode ($s = s' = s'' = 0$).

$$C^{(IV)} = -(\eta(k) + \eta(p) + \eta(q))S_{000}.$$

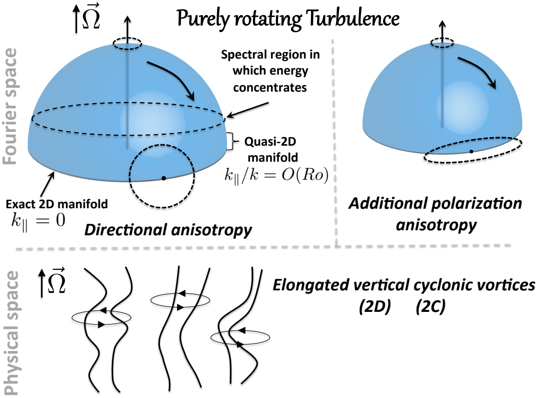
Hierarchies for statistical closures, third-order correlations at three points!



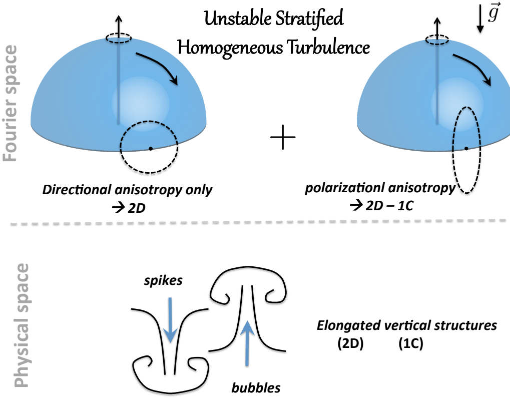
Anisotropy, disentangling directional one and polarization one, ring-to-ring vs. shell-to-shell



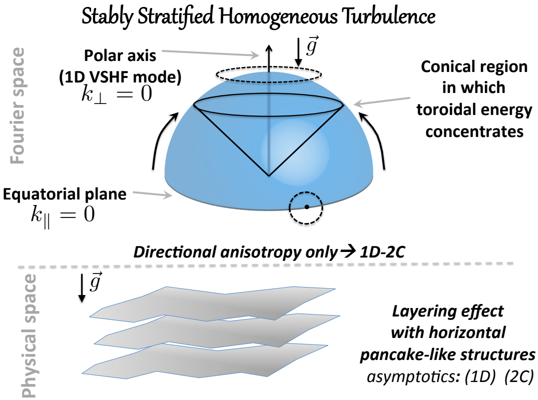
(Favier *et al.*, JFM, 2011, Sagaut & CC, 2018, Chap. 12)



(from CC & Jacquin, JFM, 1989 to Bellet *et al.*, JFM, 2006, Scott, JFM 2014, S & CC, 2018, Chap 7)

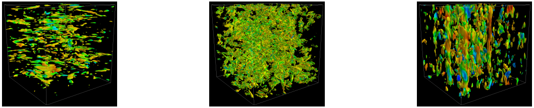


(Burlot *et al.*, JFM, PoF, 2015, S & CC, 2018, Chap 10)



(From Godeferd & CC, PoF, 1994, S & CC, 2018, Chap 10)

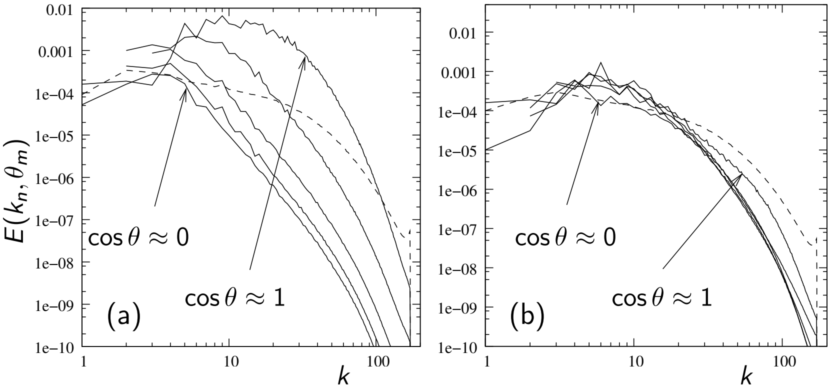
Rotating and stratified flows, anisotropic structure



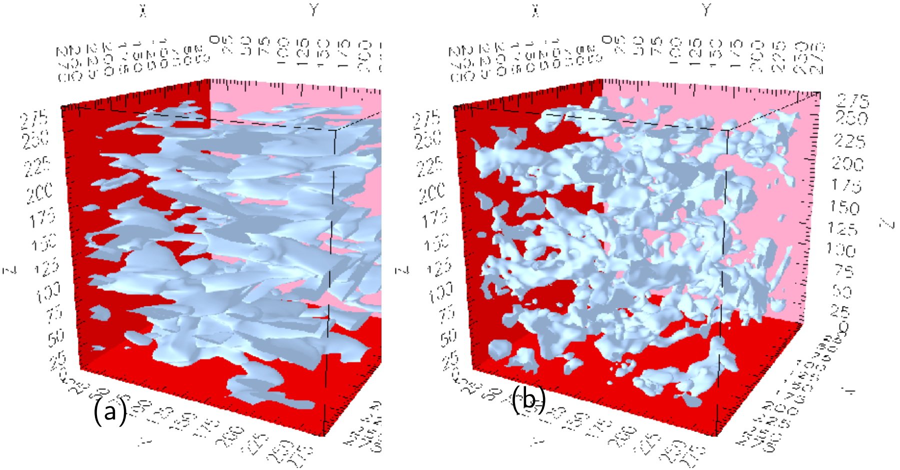
STRATIFIED $\quad 2\Omega = f = N \quad$ ROTATING

512^3 DNS from Liechtenstein *et al.* 2005. Beyond snapshots: anisotropic cascades!

Purely stably-stratified turbulence. $f = 0$, no forcing



Angle-dependent toroidal (left) and poloidal (right) modes (Liechtenstein, 2006)



X
Y
Z
(a)
(b)

The toroidal cascade and beyond

- The toroidal mode partly decouples from gravity waves. This questions a priori global scalings in terms of Froude number(s): Hanazaki & Hunt (RDT), Lindborg, Chomaz, Billand, Brethouwer, and coworkers. Coming back to Riley *et al.* (1981), with possibly *small vertical* Froude number \pause

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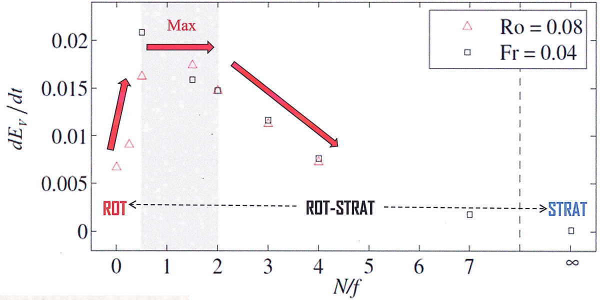
- The toroidal cascade is a 'strong' cascade, vs. a 'weak' gravity-wave turbulence cascade \pause
- It explains the layering (lasagna) even from an initially unstructured state, without need for artificial 2D horizontal forcing, or pre-existing 2D large-scale eddies (flap)

Forcing, both $N\&f$.
Evaluations of inverse cascades

From Marino *et al.* 2013, 2014.

Inverse cascade: fast growth of the large scales

$\frac{1}{2} \leq N/f \leq 2$ $\rightarrow$ No wave resonces
Smith and Waleffe, JFM, vol. 451, 145 (2002)



Inertial-gravity waves

$$\sigma(k) = k^{-1}\sqrt{N^2 k_\perp^2 + f^2 k_\parallel^2}$$

for $N/f = 1 \rightarrow \omega_k = \pm N \rightarrow \cancel{\omega_k + \omega_p + \omega_q = 0}$

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Marino et al., EPL, vol. 102, 44006 (2013)

Weakly compressible homogeneous isotropic turbulence

- A rather old study, at least in my lab. from 1988 to 1997, three successive Ph D students, J.D. Marion, F. Bataille, G. Fauchet, with J. P. Bertoglio.
- New insight for writing our books with P. Sagaut, 2008, 2018.
- Serious restart in 2017, with A. Briard, CC and P. Sagaut.

A simplified model of (quasi-isentropic) equations

$$\frac{\partial u'_i}{\partial t} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x_i} - \nu\frac{\partial u'_i}{\partial x_k \partial x_k} - \frac{\nu}{3}\frac{\partial}{\partial x_i}\left(\frac{\partial u'_k}{\partial x_k}\right) = -u'_j\frac{\partial u'_i}{\partial x_j} \tag{1}$$

$$\frac{\partial}{\partial t}\left(\frac{p'}{\gamma P}\right) + \frac{\partial u'_i}{\partial x_i} = -u'_j\frac{\partial}{\partial x_j}\left(\frac{p'}{\gamma P}\right) \tag{2}$$

- Quasi-isentropic because dissipative terms are kept for mathematical and numerical convenience
- Nonlinearity limited to second order only, fluctuation of density and pressure are implicitely small with respect to mean reference values. Mach number implicitely small too. $c_0^2 = \gamma\frac{P}{\bar{\rho}}$

Use of the fully spectral decomposition

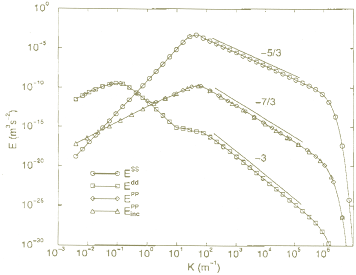
$u^{(4)} = \imath \frac{\hat{p}}{\bar{\rho} c_0}$, + the three-component toro-polo (solenoidal) - dilatational (Craya-Herring)

$$\frac{d}{dt}\begin{pmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \\ u^{(4)} \end{pmatrix} + \begin{pmatrix} \nu k^2 & 0 & 0 & 0 \\ 0 & \nu k^2 & 0 & 0 \\ 0 & 0 & \frac{4}{3}\nu k^2 & -c_0 k \\ 0 & 0 & c_0 k & 0 \end{pmatrix}\begin{pmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \\ u^{(4)} \end{pmatrix} = \begin{pmatrix} T_{NL}^{(1)} \\ T_{NL}^{(2)} \\ T_{NL}^{(3)} \\ T_{NL}^{(4)} \end{pmatrix} \tag{3}$$

where all nonlinear terms (right-hand-sides) as follows:

$$\begin{pmatrix} T_{NL}^{(1)} \\ T_{NL}^{(2)} \\ T_{NL}^{(3)} \\ T_{NL}^{(4)} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{e}^{(1)} \cdot \left(\widehat{\boldsymbol{\omega} \times \boldsymbol{u}'}\right) \\ -\boldsymbol{e}^{(2)} \cdot \left(\widehat{\boldsymbol{\omega}' \times \boldsymbol{u}'}\right) \\ -\boldsymbol{e}^{(3)} \cdot \widehat{\boldsymbol{\omega}' \times \boldsymbol{u}'} - \frac{1}{2}\imath k \widehat{u_j' u_j'} \\ \imath u_j \widehat{\frac{\partial (p'/(\bar{\rho} a_0))}{\partial x_j}} \end{pmatrix} \tag{4}$$

Main results about spectra of the nonlinear quasi-isentropic model, Fauchet etal. 1997



Fauchet et al. (1997), using EDQNM/DIA/DNS.

- Acoustic equilibrium only at very small k.
- Classical (as in solenoidal) pressure spectrum (Batchelor) for other k's.

Table: Two-point closure prediction dealing with inertial range in the low-Mach number régime ($M_t < 0.1$).

Decorrelation function	$\sim \exp[-\eta(k)(t-t')]$	$\sim \exp[-\eta^2(k)(t-t')^2]$
$E_{dd}(k)$	$\propto M_t^2 Re_L^1 k^{-11/3}$	$\propto M_t^4 Re_L^0 k^{-3}$
$E_{pp}(k)$	$\propto M_t^2 Re_L^1 k^{-11/3}$	$\propto M_t^2 Re_L^0 k^{-7/3}$
$E_{p'p'}^{acous}(k)$	$\sim E_{dd}(k)$	$\propto M_t^6 Re_L^0 k^{-11/3}$
$\lim_{M_t \to 0} E_{pp}(k)$	$\neq E_{pp}^{inc}(k)$	$= E_{pp}^{inc}(k)$
k_d/k_s	$\propto M_t^2 Re_L^1$	$\propto M_t^4 Re_L^0$
$\bar{\varepsilon}_d/\bar{\varepsilon}_s$	$\propto M_t^2 Re_L^0$	$\propto M_t^4 Re_L^{-1} \ln(Re_L)$

Achievements and remaining challenges

- Strategy EDQNM2 applied with all details using the acoustic dispersion frequency $\sigma_k = c_0 k$ and resonance operator $\exp(\imath c_0(sk + s'p + s"q)(t - t'))$ coupled with a ED factor, especially needed for $s = 0$. Derivation much clearer than in previous studies, advocating DIA, but problems remain
- Recovering the strict incompressible limit, with correct M_t law?
- Need for a Gaussian rather than an exponential decorrelation function?
- A possible new interpretation of η in the Gaussian kernel: a standard variation for $c_0 k$ (partly random, as in the kraichnan's random oscillator) but without renormalize the laminar viscosity, nor the mean value of the sonic speed.
- Taking into account mass-averaged energy? $\rho_0 uu \rightarrow \rho uu$

Generalized EDQNM and beyond?

- A general strategy, not a new theory, equations not carved in the marble. To be matched with Wave-Turbulence theory.
- Possibility to take into account detailed anisotropy, including directional one connected to dimensionality , from 3D to 2D, 1D. Effects of mean gradients, body forces: not a perturbative approach, without formal expansion around isotropy as in (Kraichnan's legacy, DIA, LHDIA, TFM, LRA ... etc)
- Fully numerical solution, with quantitative comparison with DNS at highest resolution (CC et al., JFM 1997, Burlot et al., JFM 2015) Integration over the orientation of triads: fully numerical (from CC & Jacquin 1989, Bellet *et al.* 2006) to semi-analytical (but with truncated anisotropy) with Mons *et al.* 2016.
- An unprecedented investigation of the finite Reynolds number effect, initial data, parametric study in general.

Observations, tools

- Is conventional pseudo-spectral DNS in tri-periodic box the best tool? Discretization of smallest scales (e.g. $k* = 1, 10$), capture of slow manifolds, especially in that range (e.g. $k_{||}$ =0, 2D) \pause

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- Some pictures and non-conventional proposals, EDQNM for a *supergrid* model? \pause
- Better numerical resolution, for anisotropy, infrared range, what about *internal intermittency*. Is it really an objective syndrom (or symptom)?