The 3-waves collision term in condensed Bose gases

M. Escobedo

Dept. of Mathematics

Universidad del País Vasco, Bilbao

(Joint work with E. Cortés, BCAM, Bilbao.)

Wave Interactions and Turbulence, Ecole Polytechnique, December 2018

The 3-excitations collision term

$$C_{1,2}(n) = \iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \left(R(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) - R(\mathbf{k}_1,\mathbf{k},\mathbf{k}_2) - R(\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}\right) d^3\mathbf{k_1}\mathbf{d^3}\mathbf{k_2}$$

$$R(\mathbf{k},\mathbf{k_1},\mathbf{k_2}) = |\mathcal{M}(\mathbf{k},\mathbf{k_1},\mathbf{k_2})|^2 \, \delta\left(\omega(\mathbf{k}) - \omega(\mathbf{k_1}) - \omega(\mathbf{k_2})\right) \times$$
$$\times \, \delta\left(\mathbf{k} - \mathbf{k_1} - \mathbf{k_2}\right) \left(n_1 n_2 (1+n) - (1+n_1)(1+n_2)n\right)$$

$n(t,\mathbf{k})$ density of excitations at time t and momentum $\mathbf{k}$; $n_1(t) \equiv n(t,\mathbf{k}_1)$, ...

$\omega(\mathbf{k})$ is the energy of excitations of momentum $\mathbf{k}$

$|\mathcal{M}(\mathbf{k},\mathbf{k_1},\mathbf{k_2})|^2$ is the scattering amplitude.

In a condensed Bose gas:

Number-changing processes between superfluid component and the normal fluid (excitations).

The collision integral $C_{1,2}$ describes $1 \leftrightarrow 2$ splitting of an excitation into two others in the presence of the condensate.

T. R. Kirkpatrick and J. R. Dorfman in several articles, PRA 1983, (JLTP 1985)[3] derived the kinetic equation in a uniform Bose gas which includes these processes.

Similar collision integral for different $\omega(\mathbf{k})$ and $|\mathcal{M}(\mathbf{k}, \mathbf{k_1}, \mathbf{k_2})|^2$ as in:
R. E. Peierls '29 (cristal lattices), D. J. Benney & P. G. Saffman '65 (random waves in dispersive medium), V. E. Zakharov '65 (capillary waves), many examples in V. E. Zakharov's & al. "Kolmorov Spectra of Turbulence Turbulence " '92,...

The case of the gas of bosons was considered in detail by S. Dyachenko & al. Phys. D'92 for a general class of Hamiltonian systems.

The question has also been treated in:
D. V. Semikoz & al.'95; Y. Pomeau & al.'99; R. Lacaze & al.'01; C. Connaughton & Y. Pomeau'04; Ch. Josserand & al.'08, ...

Described mathematical properties of these equations such as: derivation, Kolmogorov-Zakharov solutions and their stability, long time self similar behavior...

Some recent results in the maths literature:

M.E. & E. Cortés (ArXiv '18)

R. Alonso, I.M. Gamba & M.B. Tran (ArXiv'18)

For a spatially homogeneous condensed Bose gas, a system may be written as:

$$\frac{\partial n}{\partial t}(t, \mathbf{k}) = C_{1,2}(n)(t, \mathbf{k}),$$
$$\frac{dn_c(t)}{dt} = -\int_{\mathbb{R}^3} C_{1,2}(n)(t, \mathbf{k})\, d^3\mathbf{k}$$

with $n_c = n_c(t)$: condensate density.

This system formally ensures conservation of number of particles and energy.

The excitations density $n(t)$ may be a measure, but the description assumes:

$$n(t, \{0\}) = 0 \;\; for \; all \; t > 0.$$

The dispersion law is: $\omega(k) = \sqrt{\frac{gn_c}{m}k^2 + \left(\frac{k^2}{2m}\right)^2}$

where $g = 4\pi a/m$ and a is the s-wave scattering length.

If we denote: N the total particle density and λ: thermal de Broglie wavelength. Two different regions of the parameters λ, n, a are usually considered (Kirkpatrick & al. '85; Dyachenko & al. '92;...):

- $N a \lambda^2 << 1, \ N\lambda^3 \geq 1:$ the "moderately low temperature region"
- $N a \lambda^2 \geq 1:$ the "low temperature region".

In the first case: $\omega(k) = \frac{k^2}{2m}, \qquad |\mathcal{M}(\mathbf{k}, \mathbf{k_1}, \mathbf{k_2})|^2 = \frac{8 n_c a^2}{m^2}$

For an isotropic gas:

$$n(t, \mathbf{k}) = f(t, x), \ x = |\mathbf{k}|^2; \ \ g(t, x) = \sqrt{x}\, f(t, x)$$

The system seems to be (after some scaling in time to absorb constants):

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,x) = \ n_c(t)Q(g,g) \\ \qquad n_c'(t) = -\, n_c(t) \displaystyle\int_0^\infty Q(g,g)(t,x)dx \end{cases}$$

$$Q(g,g) = \int_0^x \left(\frac{g(y)g(x-y)}{\sqrt{y}\sqrt{x-y}} - \frac{g(x)}{\sqrt{x}} \left[\frac{g(x-y)}{\sqrt{x-y}} + \frac{g(y)}{\sqrt{y}} \right] \right) dy +$$

$$+\, 2 \int_x^\infty \left(\frac{g(y)}{\sqrt{y}} \left[\frac{g(y-x)}{\sqrt{y-x}} + \frac{g(x)}{\sqrt{x}} \right] - \frac{g(y-x)}{\sqrt{y-x}} \frac{g(x)}{\sqrt{x}} \right) dy - \sqrt{x}\, g(x) + 2 \int_x^\infty \frac{g(y)}{\sqrt{y}} dy$$

- The point is: g may be singular at the origin like the equilibria $\dfrac{\sqrt{x}}{e^{\beta x} - 1}$.

In general, if $g(x) \sim \frac{1}{\sqrt{x}}$ near zero, $Q(g,g)$ does not converge.

We first want to understand: what does the system actually means?

Start from the original system for $(n(t, \mathbf{k}), n_c(t))$

$\rightarrow$ define precisely what is a weak solution for $n(t, \mathbf{k}) + n_c(t)\delta^3(0)$ where:

- $n(t)$ is a non negative measure such that $n(t, \{0\}) = 0$.
- Use test functions φ such that "see $x = 0$": $\varphi(0) > 0$

Then take radial test functions for radial $n(t, \mathbf{k}) = f(t, x)$ and $g = \sqrt{x}\, f$.

If we denote the measure: $g(t, x) + n_c(t)\delta(0) = G(t, x)$ the weak formulation is:

for all $\varphi \in C_b^2([0,\infty))$,

$$\frac{d}{dt}\int_{[0,\infty)} \varphi(x) G(t,x)dx = n_c(t)\Bigg(\iint_{(0,\infty)^2} \frac{\Lambda_\varphi(x,y)}{\sqrt{xy}} g(t,x)g(t,y)dxdy + \\ + \int_{(0,\infty)} \frac{L_\varphi(x)}{\sqrt{x}} g(t,x)dx \Bigg)$$

$$\Lambda_\varphi(x,y) = \varphi(x+y) + \varphi(|x-y|) - 2\varphi(\max\{x,y\})$$

$$L_\varphi(x) = 2\int_0^x \varphi(y)dy - x(\varphi(x) + \varphi(0))$$

All these integrals are now <u>absolutely convergent</u> .

If we denote: $M_{1/2}(g) = \displaystyle\int_0^\infty \sqrt{y}\, g(y)dy$:

Result 1. For a non negative measure $G(t) = n_c(t)\delta_0 + g(t)$, with $g(t,\{0\}) \equiv 0$, to be a solution of the weak formulation is equivalent to:

1. $$\frac{\partial g}{\partial t}(t) = n_c(t) Q(g(t), g(t)) \quad \boxed{in \;\; \mathscr{D}'(0,\infty),} \quad for\, all\;\; t > 0$$

$$where: \qquad Q(g,g) = \int_0^x \left(\frac{g(y)g(x-y)}{\sqrt{y}\sqrt{x-y}} - \frac{g(x)}{\sqrt{x}} \left[\frac{g(x-y)}{\sqrt{x-y}} + \frac{g(y)}{\sqrt{y}} \right] \right) dy +$$
$$+ 2\int_x^\infty \left(\frac{g(y)}{\sqrt{y}} \left[\frac{g(y-x)}{\sqrt{y-x}} + \frac{g(x)}{\sqrt{x}} \right] - \frac{g(y-x)}{\sqrt{y-x}} \frac{g(x)}{\sqrt{x}} \right) dy -$$
$$- \sqrt{x}\, g(x) + 2\int_x^\infty \frac{g(y)}{\sqrt{y}} dy.$$

2. $$n_c(t) - n_c(0) + \int_0^t n_c(s) M_{1/2}(g(s)) ds = \mu_{n_c,g}((0,t])$$

The “flux term” $\mu_{n_c,g}$

$\mu_{n_c,g}$ is a non negative measure such that:

$$\mu_{n_c,g}((0,t]) = \lim_{\varepsilon\to 0}\int_0^t n_c(s)\left(\iint_{(0,\infty)^2}\frac{\Lambda_{\varphi_\varepsilon}(x,y)}{\sqrt{xy}}g(t,x)g(t,y)dxdyg(s)\right)ds$$

$$\Lambda_{\varphi_\varepsilon}(x,y) = \varphi_\varepsilon(x+y) + \varphi_\varepsilon(|x-y|) - 2\varphi_\varepsilon(\max\{x,y\})$$

$$\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$$

for any convex, non negative function $\varphi \in C_b^1([0,\infty))$ such that:

$$\varphi(0) = 1 \quad and \qquad \lim_{x\to\infty}\sqrt{x}\varphi(x) = 0.$$

- Using the Result 1, one may check that

$$G = C\delta_0 + \frac{\sqrt{x}}{e^{\beta x} - 1}$$

is a weak solution for all constants $\beta > 0$, $C \geq 0$.

- The term $\mu_{n_c,g}$ is related with the behavior of g at $x = 0$:

\- A. Nouri'07. If g is $L^1(0,\infty)$ and $x = 0$ is a Lebesgue point, then for all $n_c > 0$, $\mu_{n_c,g} \equiv 0$.

\- H. Spohn'10. If

$$g(x) \sim \frac{a}{\sqrt{x}}, \ \ as\ x \to 0,$$

for some $a > 0$, and $\int_0^\infty \sqrt{x} g(x) dx < \infty$ then,

$$\mu_{n_c,g}([0,t) = -\left(\frac{\pi^2}{3}a^2 + \int_0^\infty \sqrt{x} g(x) dx\right) t$$

Property 1. For all initial data (n_0, g_0), where g_0 is a non negative measure such that $\int_{(0,\infty)} xg(x)dx < \infty$ and $m_0 > 0$, we prove the existence of a solution $(n_c(t), g(t))$, such that: $n_c(0) = n_0$, $g(0, x) = g_0(x)$, and the total number of "particles" and the energy are conserved. Moreover: $\underline{\mu([0, t)) > 0 \text{ for all } t > 0}$.

For all initial data a non negative measure: the flux $\mu([0, t))$ is instantaneously and always strictly positive.

The exact behavior of g near the origin $x = 0$ is not known. But:

Property 2. If $G(t)$ is a weak solution without atoms, such that

$$\int_{(0,\infty)} G(t,x)x^{-1/2}dx < \infty \ \ for \ \ t \in (0,T),$$

then $\mu([0,t)) = 0$ for t in $(0,T)$.

Property 3. For all $T > 0$, $R > 0$ and $\alpha \in \left(-\frac{1}{2}, \infty\right)$,

$$\int_0^T n_c(t)) \int_{(0,R]} x^\alpha g(t,x)dxdt \leq$$

$$\leq \frac{2R^{\frac{1}{2}+\alpha}}{1-\left(\frac{2}{3}\right)^{\frac{1}{2}+\alpha}} \left(\int_0^T n_c(t)dt\right)^{\frac{1}{2}} \left(\frac{\sqrt{E}}{2}\int_0^T n_c(t)dt + \sqrt{N}\right).$$

Consequence. If $G(t) = \alpha(t)\delta_0 + g(t)$ and g has no atoms:

$$\int_0^T n_c(t)) \int_{(0,\infty)} x^\alpha g(t,x)dxdt < \infty,\ for\ all\ \alpha > -1/2$$

$$\int_{(0,\infty)} x^{-1/2} g(t,x)dx = \infty,\ \ for\ all\ t > 0.$$

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- Similar properties as for the equilibria $\dfrac{\sqrt{x}}{e^{\beta x} - 1}$.
- The collision integral $Q(g,g)$ does not convrege.

The condensate density $n_c(t)$ decreases due to the term $M_{1/2}(g(t)$ but increases due to the flux:

$$n_c(t) = n_c(0) - \int_0^t n_c(s) M_{1/2}(g(s)) ds + \mu_{n_c,g}((0,t])$$

Then, $n_c(t)$ may be not monotone decreasing since

However, if the number of particles N and the energy E are such that:

$$\frac{E}{N^{5/3}} > 6^{2/3}.$$

Then, $n_c(t) \to 0$ as $t \to \infty$ and fast enough in order to:

$$\int_0^\infty n_c(t) dt < \frac{3N^{\frac{3}{2}} M_2(0)}{E(E^{3/2} - 6N^{5/2})}.$$

In the case: $Na\lambda^2 \geq 1$ (low temperature region)

$$\omega(\mathbf{k}) = c\mathbf{k} = ck, \; |\mathcal{M}(\mathbf{k}, \mathbf{k_1}, \mathbf{k_2})|^2 = \frac{9\, c\, kk_1k_2}{64\pi^2 m\, n_c^2}, \;\; c = 2\sqrt{\frac{\pi a n_c}{m^2}}$$

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for $n(t, \mathbf{k}) \equiv n(t, k)$:

$$\begin{aligned} \frac{\partial n}{\partial t}(t, k) \;\; = \;\; & \frac{1}{n_c(t)} \int_0^k k^2 k'^2 (k - k')^2 \left[n(t, k')n(t, k - k') - n(t, k')n(t, k) - \right. \\ & \left. - n(t, k - k')n(t, k) - n(t, k) \right] dk' - \\ & -2 \int_0^\infty k^2 k'^2 (k + k')^2 \left[n(t, k)n(t, k') - n(t, k')n(t, k + k') - \right. \\ & \left. - n(t, k)n(t, k + k') - n(t, k + k') \right] dk' \end{aligned}$$

Due to the kernel: $kk'(k - k')$ and $kk'(k + k')$: problem is regular provided:

(i) no (too) fat tails ($\rightarrow$ no problem).

(ii) $n_c(t) > 0$ The equation has such a mechanism. If, for some $\delta > 0$:

$$n_c(0) \geq C_0 - M_2(0) + \delta,$$

then, $\underline{n_c(t) \geq \delta}$ for all $t > 0$, where:

$$C_0: \text{ explicit positive constant depending on } M_3(0), M_4(0) \text{ and } \sup_{k \geq 0} \left(|k|^2 n(0,k)\right).$$

$$M_\rho(0) = \int_0^\infty k^\rho n(0,k) dk$$

The solutions are such that all the integrals in the equation converge absolutely.