Gravitational wave turbulence in the primordial universe

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Observation of Gravitational Waves from a Binary Black Hole Merger

B. P. Abbott *et al.**
(LIGO Scientific Collaboration and Virgo Collaboration)
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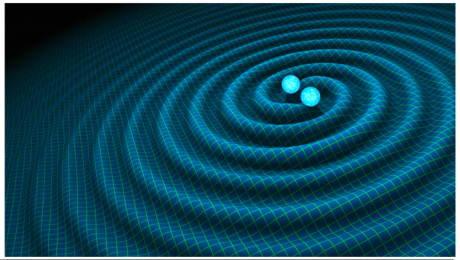
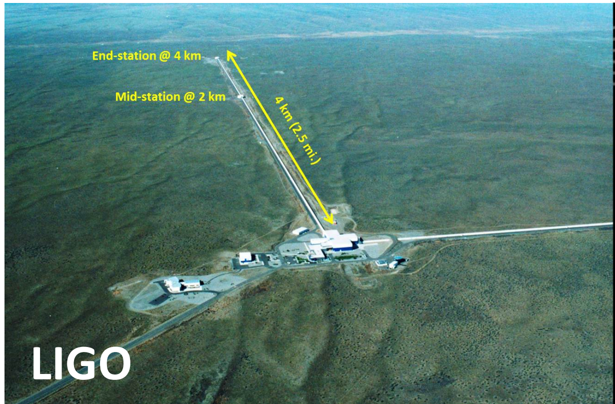
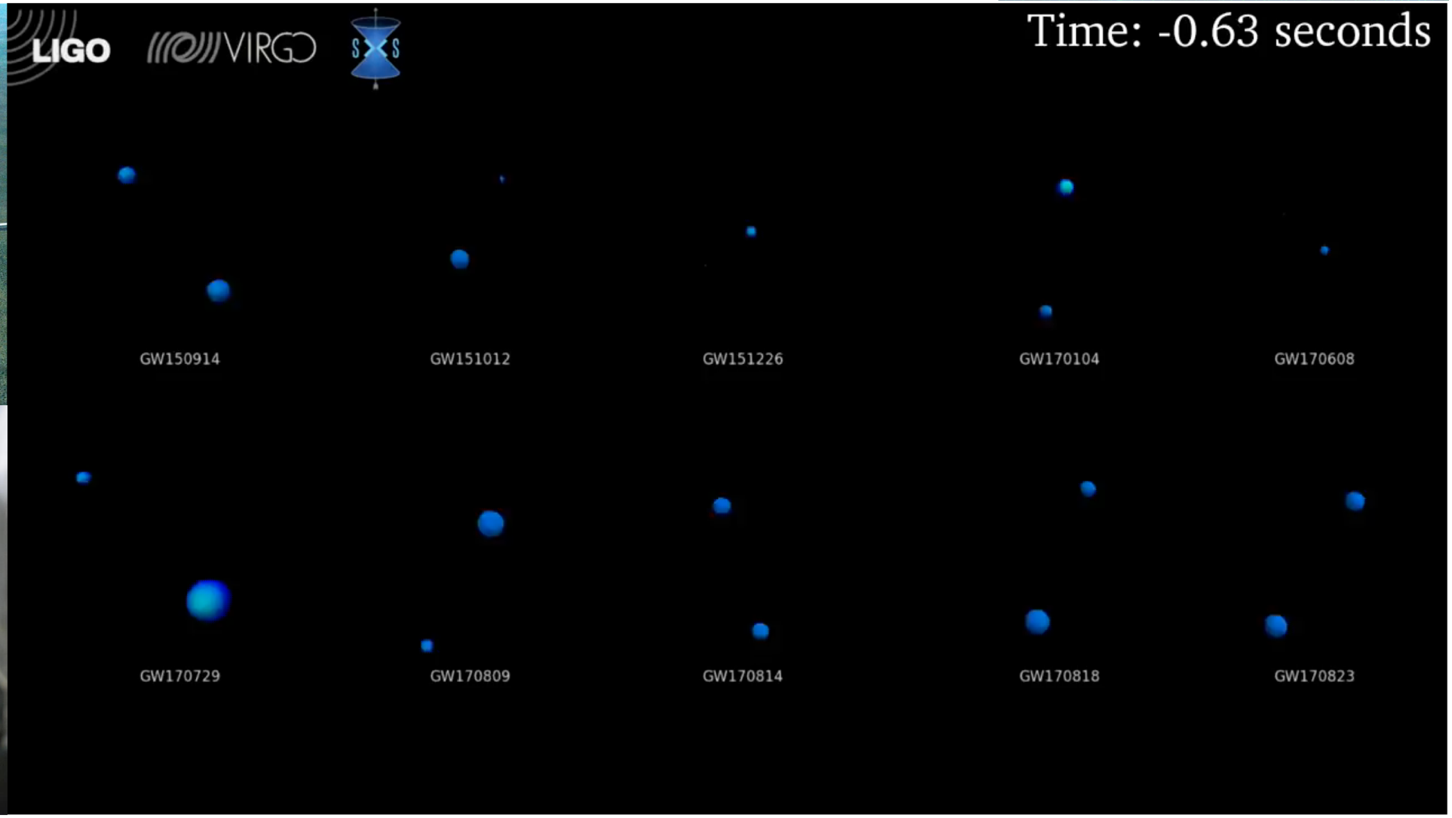
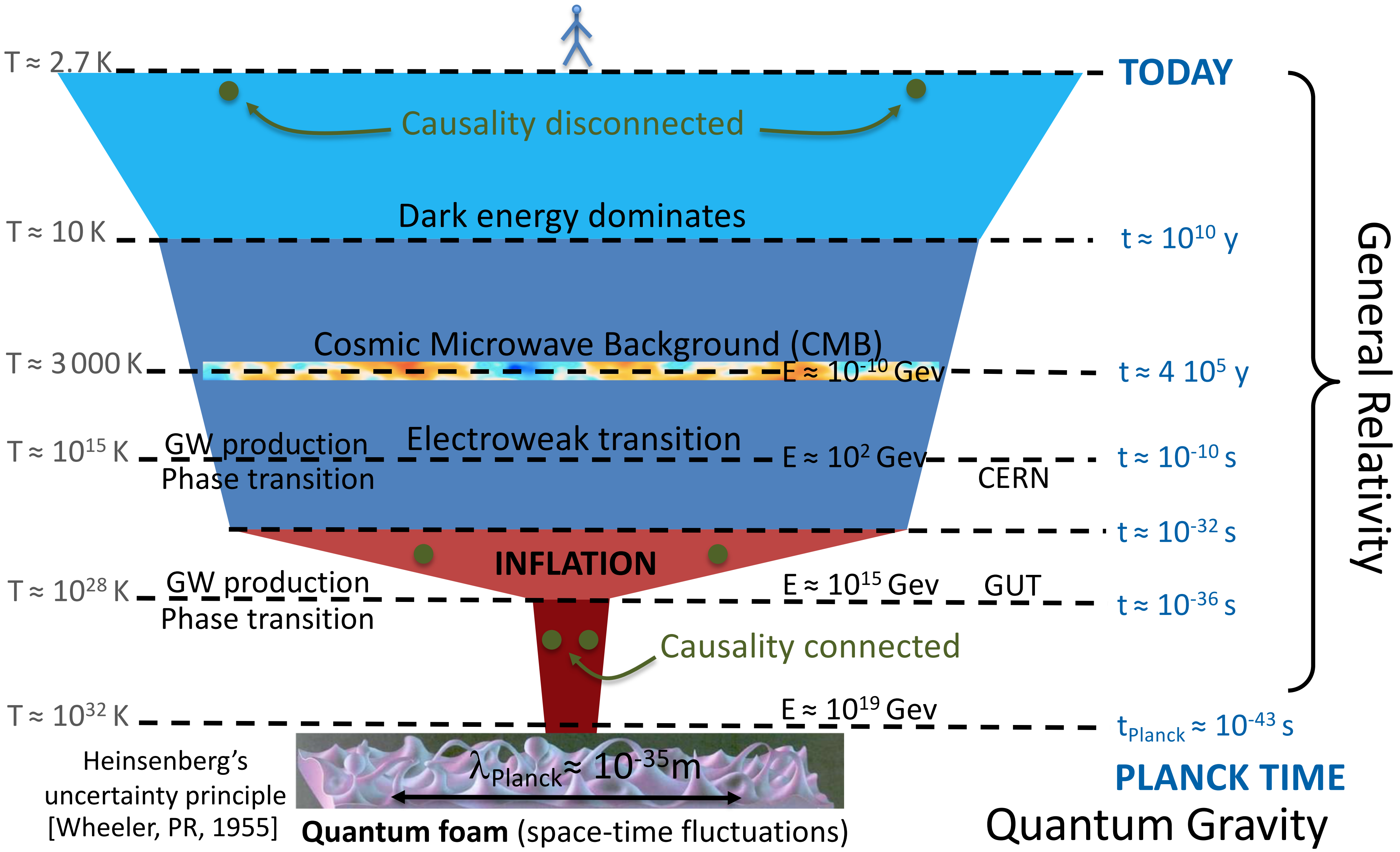






4 new detections [arXiv:1811.12907v1]

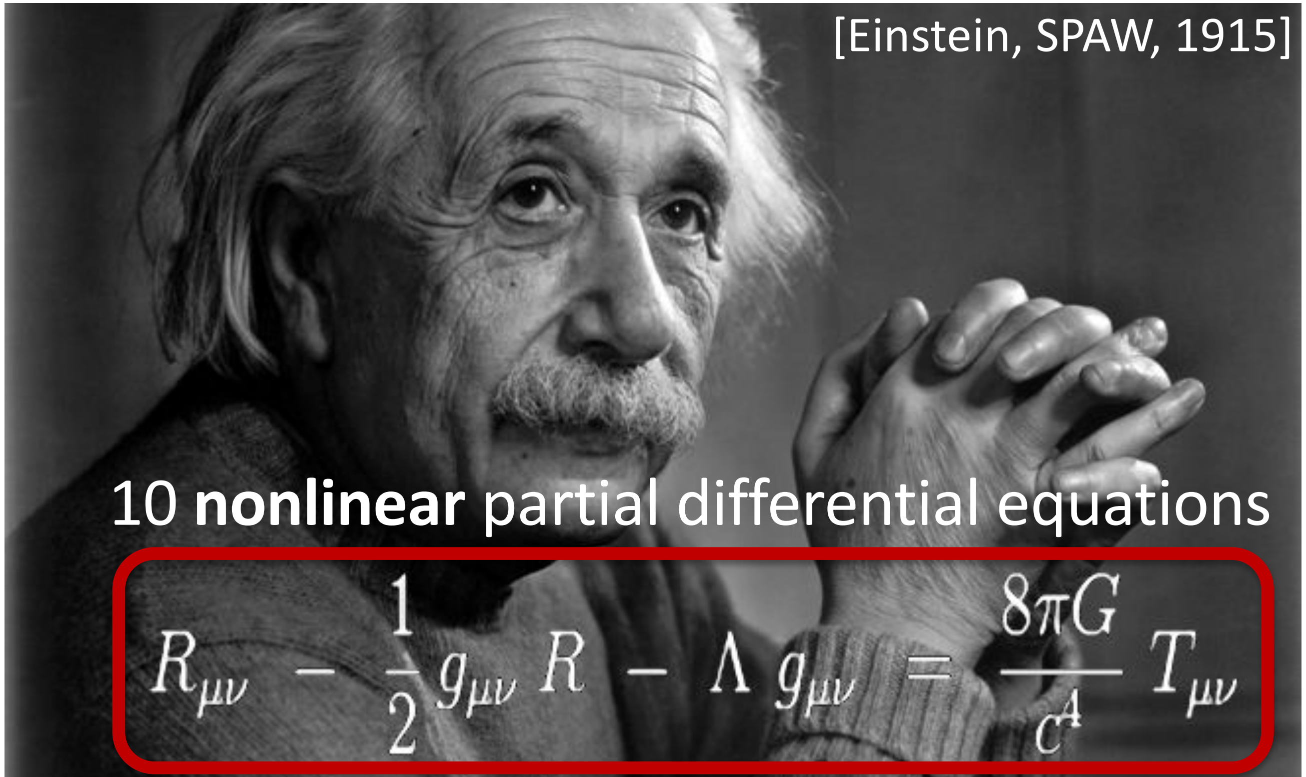
History of the Universe

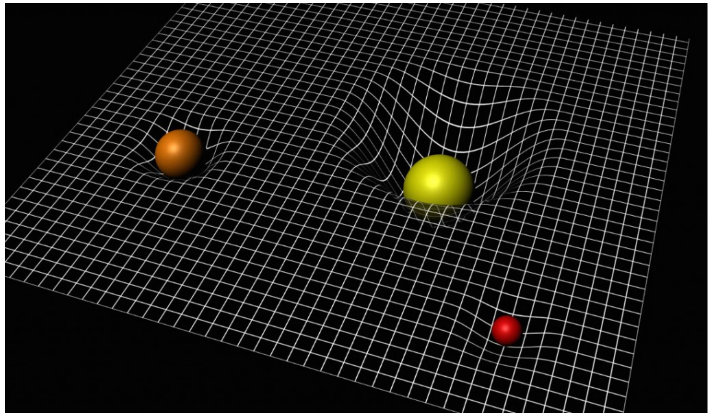


Inflation is explained by introducing an hypothetical scalar field called *inflaton* [Guth, 1981]

[see however eg: Ijjas+, PLB, 2013]

Einstein equations





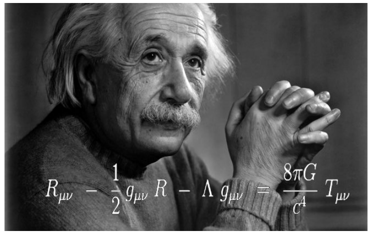
$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left[\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}\right]$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

- $R_{\mu\nu}$: Ricci tensor
- R : Ricci scalar
- $g_{\mu\nu}$: metric tensor
- Λ : cosmological constant

- $\Gamma^\alpha_{\mu\nu}$: Christoffel symbol
- $T_{\mu\nu}$: stress-energy
- $G = 6.67\ 10^{-11}\ m^3 kg^{-1} s^{-2}$
- $c = 2.99\ 10^8\ ms^{-1}$



Gravitational waves

$\Lambda = 0$

Exact linear solutions in an empty – flat – Universe: $\boxed{R_{\mu\nu} = 0}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } h_{\mu\nu} \ll 1$$

Poincaré-Minkowski
space-time metric

Effect of a + gravitational wave
on a ring of particles
(h = 0.5)

$$\omega_{\mathbf{k}} = c|\mathbf{k}| = ck$$

$$h^{+}_{\mu\nu} = a \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Weakly nonlinear general relativity $\Lambda=0$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } h_{\mu\nu} \ll 1 \qquad R_{\mu\nu} = 0$$

$$R_{\mu\nu} = R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \cancel{R^{(3)}_{\mu\nu} + R^{(4)}_{\mu\nu} + \ldots} \qquad R^{(1)}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu}$$

<u>Triadic interactions:</u> $\begin{cases} \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \text{ and } \omega_{\mathbf{k}} = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} \\ \omega_{\mathbf{k}} = c|\mathbf{k}| = ck \end{cases}$

$\Rightarrow$ **Collinear wave vectors**

We found no contribution on the **resonant manifold**

Three-wave interactions in GW turbulence **does not contribute**!

[SG & Nazarenko, PRL, 2017]

Weakly nonlinear general relativity $\Lambda=0$

Einstein-Hilbert action: $S = \frac{1}{2}\int R\sqrt{-g}\,d^4x$

g is the determinant of $g_{\mu\nu}$

R is the scalar curvature

Diagonal space-time metric:

$$\boxed{\partial/\partial z = 0}$$

$$g_{\mu\nu} = \begin{pmatrix} -(H_0)^2 & 0 & 0 & 0 \\ 0 & (H_1)^2 & 0 & 0 \\ 0 & 0 & (H_2)^2 & 0 \\ 0 & 0 & 0 & (H_3)^2 \end{pmatrix}$$

[Hadad & Zakharov, JGP, 2014]

$$H_0 = e^{-\lambda}\gamma, \quad H_1 = e^{-\lambda}\beta, \quad H_2 = e^{-\lambda}\alpha, \quad H_3 = e^{\lambda}$$

(Lamé coefficients)

Lagrangian density:

$$\Rightarrow \quad \mathcal{L} = \frac{1}{2}\left[\overbrace{\frac{\alpha\beta}{\gamma}\dot{\lambda}^2 - \frac{\alpha\gamma}{\beta}(\partial_x\lambda)^2 - \frac{\beta\gamma}{\alpha}(\partial_y\lambda)^2}^{\text{Give the linear contribution}} - \frac{\dot{\alpha}\dot{\beta}}{\gamma} + \frac{(\partial_x\alpha)(\partial_x\gamma)}{\beta} + \frac{(\partial_y\beta)(\partial_y\gamma)}{\alpha}\right]$$

$$\alpha = \beta = \gamma = 1 \qquad \lambda \ll 1 \qquad \lambda = c_1\exp(-i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x}) + c_2\exp(i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{x})$$

Hadad & Zakharov's theorem (JGP, 2014)

- Dynamical equations given by: $$\begin{cases} \dfrac{\delta S}{\delta \lambda} = 0 \\ \dfrac{\delta S}{\delta \alpha} = \dfrac{\delta S}{\delta \beta} = \dfrac{\delta S}{\delta \gamma} = 0 \end{cases}$$ 4 equations

- Vacuum Einstein equations:

7 equations

$$[R_{\mu\nu}] = \begin{pmatrix} R_{00} & R_{01} & R_{02} & 0 \\ - & R_{11} & R_{12} & 0 \\ - & - & R_{22} & 0 \\ - & - & - & R_{33} \end{pmatrix} = [0]$$

It's compatible !

Hamiltonian formalism

Normal variables: $\lambda_{\mathbf{k}} = \frac{a_{\mathbf{k}} + a^*_{-\mathbf{k}}}{\sqrt{2k}}, \quad \dot{\lambda}_{\mathbf{k}} = \frac{\sqrt{k}(a_{\mathbf{k}} - a^*_{-\mathbf{k}})}{i\sqrt{2}},$ (Fourier space)

Hamiltonian equation: $i\dot{a}_{\mathbf{k}} = \frac{\partial H}{\partial a^*_{\mathbf{k}}}$ where $H = H_{\text{free}} + H_{\text{int}}$

$$H_{\text{free}} = \sum_{\mathbf{k}} k|a_{\mathbf{k}}|^2$$

With $R_{01}=R_{02}=R_{12}=0$ we find:

$$\begin{aligned}
H_{\text{int}} = \frac{1}{4}\sum_{1,2,3,4,5} \frac{\delta_{123}\delta^1_{45}}{\sqrt{k_2k_3k_4k_5}} \Bigg\{ \Bigg[\left(\frac{p_5}{p_1} + \frac{q_5}{q_1}\right) k_4 \left(-\frac{a_4a_5 + a^*_{-4}a^*_{-5}}{k_5+k_4} + \frac{a^*_{-4}a_5 + a_4a^*_{-5}}{k_5-k_4}\right) + \frac{p_4q_5}{p_1q_1}(a_4 + a^*_{-4})(a_5 + a^*_{-5}) \Bigg] \\
k_2k_3(a_2 - a^*_{-2})(a_3 - a^*_{-3}) + \Bigg[-\left(\frac{p_5}{p_1} - \frac{q_5}{q_1}\right)(p_2p_3 - q_2q_3)k_4 \left(-\frac{a_4a_5 + a^*_{-4}a^*_{-5}}{k_5+k_4} + \frac{a^*_{-4}a_5 + a_4a^*_{-5}}{k_5-k_4}\right) \\
+ \frac{p_4q_5}{p_1q_1}(\mathbf{k}_2\cdot\mathbf{k}_3)(a_4 + a^*_{-4})(a_5 + a^*_{-5}) \Bigg] (a_2 + a^*_{-2})(a_3 + a^*_{-3}) \Bigg\} \\
+ \frac{1}{2}\sum_{\mathbf{k},1,2,3,4} \frac{\delta^{\mathbf{k}}_{12}\delta^{\mathbf{k}}_{34}}{\sqrt{k_1k_2k_3k_4}} \Bigg\{ \frac{(\mathbf{k}\cdot\mathbf{k}_2)k_1p_3q_4}{pq} \left(-\frac{a_1a_2 + a^*_{-1}a^*_{-2}}{k_2+k_1} + \frac{a^*_{-1}a_2 + a_1a^*_{-2}}{k_2-k_1}\right)(a^*_3 + a_{-3})(a^*_4 + a_{-4}) \\
+ \frac{k_1k_3p_2q_4}{pq}\left(a_1a_2 + a_1a^*_{-2} - a^*_{-1}a_2 - a^*_{-1}a^*_{-2}\right)\left(a^*_3a^*_4 + a^*_3a_{-4} - a_{-3}a^*_4 - a_{-3}a_{-4}\right) \Bigg\}.
\end{aligned}$$

4 wave processes

Kinetic equation of GW turbulence

$$n_{\mathbf{k}} = \langle |a_{\mathbf{k}}|^2 \rangle$$

$H_{3\to1} = 0$ $\rightarrow$ Additional symmetry

$$\boxed{\dot{n}_{\mathbf{k}} = 4\pi \int |T^{\mathbf{k}\mathbf{k}_3}_{\mathbf{k}_1\mathbf{k}_2}|^2 n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_3}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} \right] \delta(\mathbf{k} + \mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)\, \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,}$$

$$\text{with } T^{12}_{34} = \tfrac{1}{4}(W^{12}_{34} + W^{21}_{34} + W^{12}_{43} + W^{21}_{43}),\ W^{12}_{34} = Q^{12}_{34} + Q^{34}_{12}$$

$$Q^{12}_{34} = \frac{1}{4\sqrt{k_1 k_2 k_3 k_4}} \left\{ 2\left(\frac{p_4}{p_1 - p_3} - \frac{q_4}{q_1 - q_3} \right) \frac{k_2(p_1 p_3 - q_1 q_3)}{k_1 - k_3} - 2\left(\frac{p_4}{p_1 - p_3} + \frac{q_4}{q_1 - q_3} \right) \frac{k_1 k_2 k_3}{k_1 - k_3} \right. \tag{12}$$

$$\left. + \left(\frac{p_2}{p_1 + p_2} - \frac{q_2}{q_1 + q_2} \right) \frac{k_1(p_3 p_4 - q_3 q_4)}{k_1 + k_2} - \left(\frac{p_2}{p_1 + p_2} + \frac{q_2}{q_1 + q_2} \right) \frac{k_1 k_3 k_4}{k_1 + k_2} + \frac{2k_1 k_3 p_2 q_4}{(p_1 + p_2)(q_1 + q_2)} + \frac{2k_1 p_3 (q_2 k_4 + k_2 q_4)}{(p_1 - p_3)(q_1 - q_3)} \right\}.$$

[SG & Nazarenko, PRL, 2017]

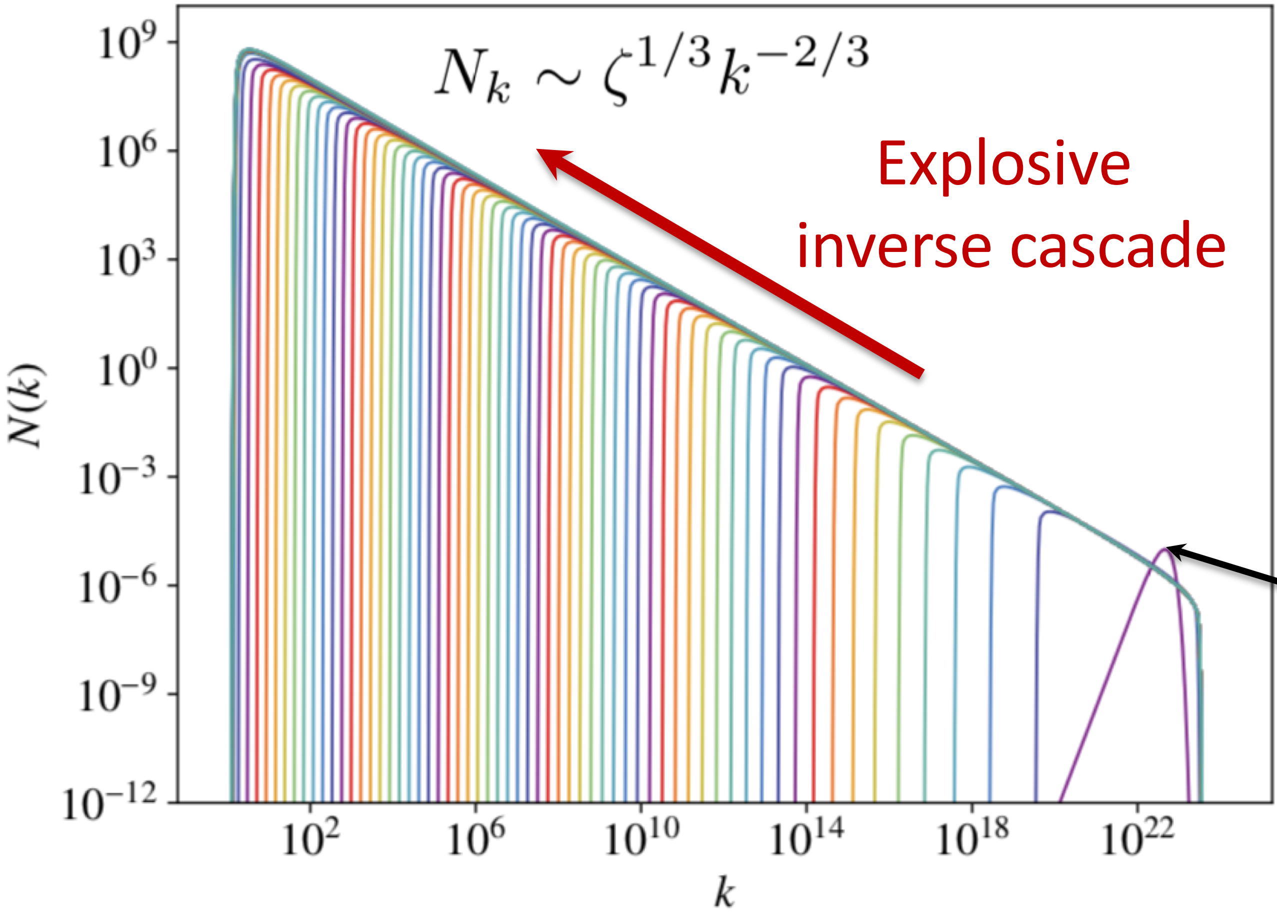
Constant flux (isotropic) spectra:

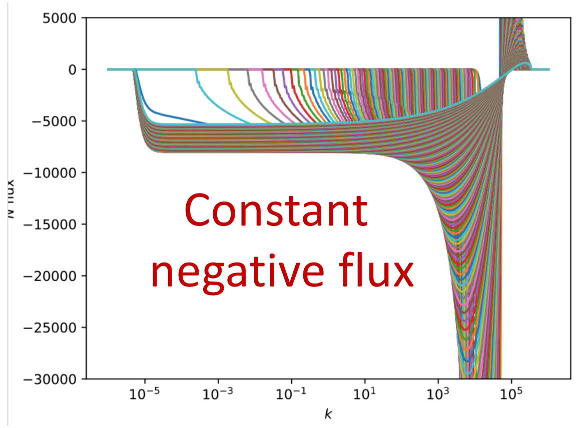
Energy	Wave action
$\mathcal{E} = \iint \omega_{\mathbf{k}} n_{\mathbf{k}} d\mathbf{k} = \text{const}$	$\mathcal{N} = \iint n_{\mathbf{k}} d\mathbf{k} = \text{const}$
$\boxed{E_k^{(1D)} \sim \varepsilon^{1/3} k^0}$	$\boxed{N_k^{(1D)} \sim \zeta^{1/3} k^{-2/3}}$ finite capacity
Direct cascade	**Inverse cascade**

Local approximation: nonlinear diffusion model

- Rigorous derivation is rare (in MHD it's possible) [SG & Buchlin, ApJ, 2010]
  [see also Dyachenko+, PD, 1992]
- Here, it's a phenomenological model

$$\frac{\partial N(k)}{\partial t} = \frac{\partial}{\partial k}\left[k^2 N^2(k)\frac{\partial (kN(k))}{\partial k}\right] - \nu k^4 N(k) - \eta \frac{N(k)}{k^4}$$

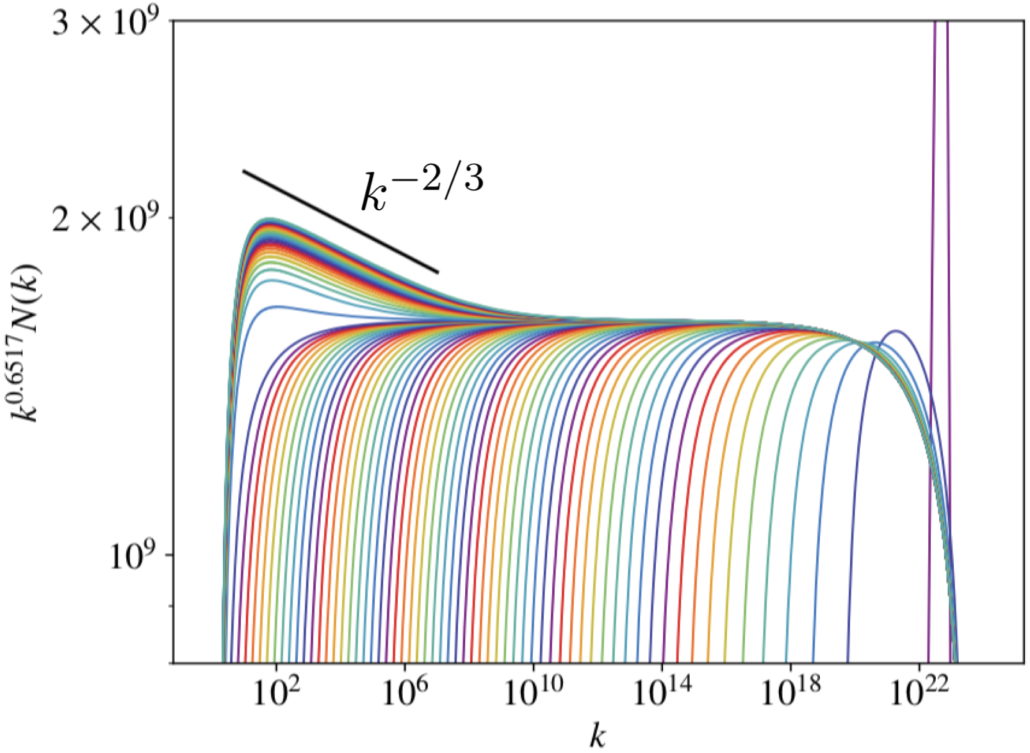




Initial spectrum
(decay simulation)

[SG, Nazarenko, Buchlin & Thalabard, 2018; arXiv:1809.07623v1]

Anomalous scaling



Old subject (weak & strong)

[Semikoz & Tkachev, PRL, 1995]

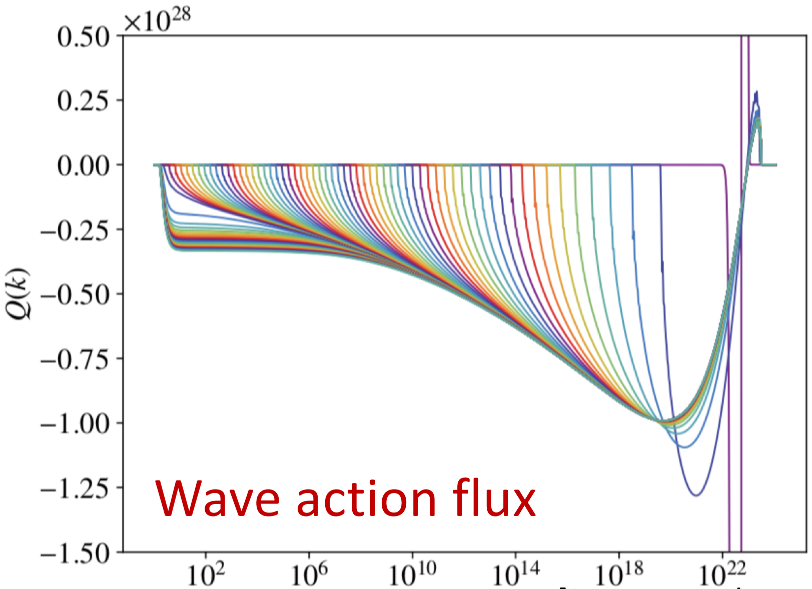
[SG+, JPP, 2000; Lacaze+, Physica D, 2001]

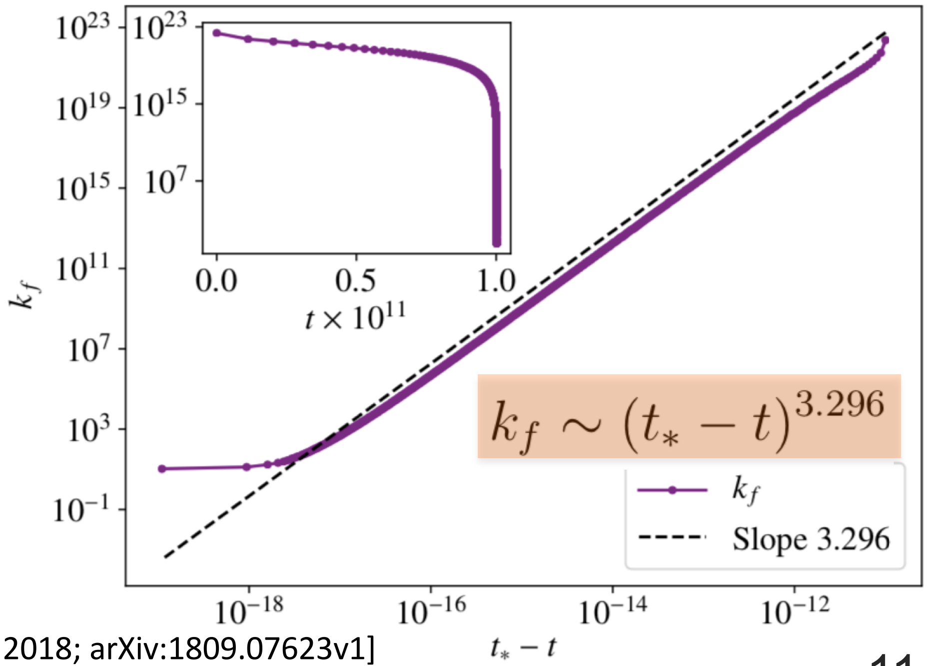
[see eg. Connaughton & Nazarenko, PRL, 2004; Nazarenko, JETPL, 2006; Boffetta+, JLTP, 2009]

[Thalabard+, JPAMT, 2015]

Self-similar solution of the second kind:

$$N = \frac{1}{\tau^\alpha} N_0 \left(\frac{k}{\tau^\beta} \right) \qquad \tau = t_* - t$$





[SG, Nazarenko, Buchlin & Thalabard, 2018; arXiv:1809.07623v1]

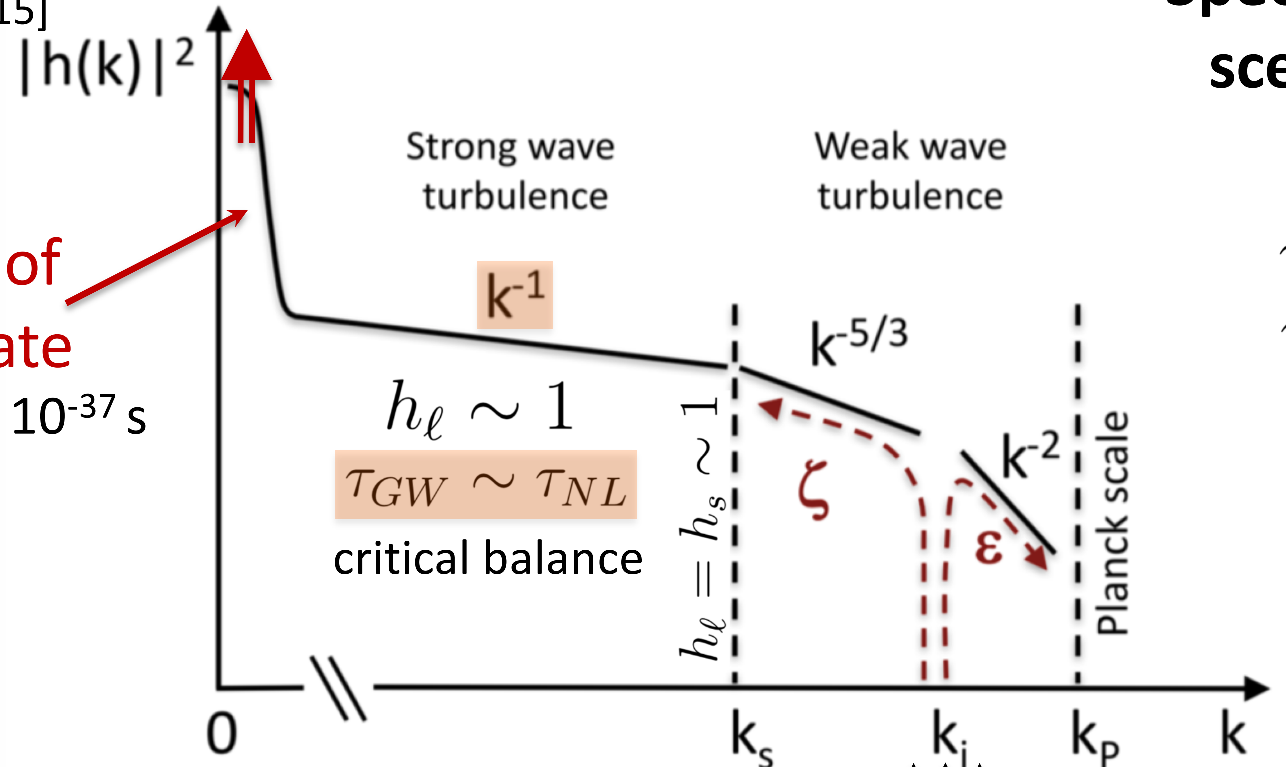
Big-Bang scenario driven by space-time turbulence ?

$t > 100\, t_{Planck} \approx 10^{-41}$ s

[SG, Laurie & Nazarenko]

[Nazarenko & Onorato, 2006; 2007]
[During+, PRE, 2015]

Speculative scenario

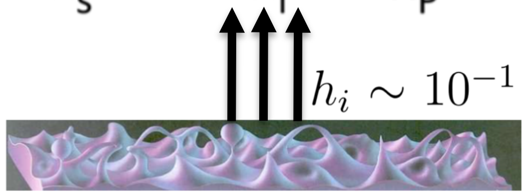


$\tau_{GW} \sim 1/\omega$
$\tau_{NL} \sim \ell/(h_\ell c)$

Formation of a condensate

$t > 10^{-37}$ s

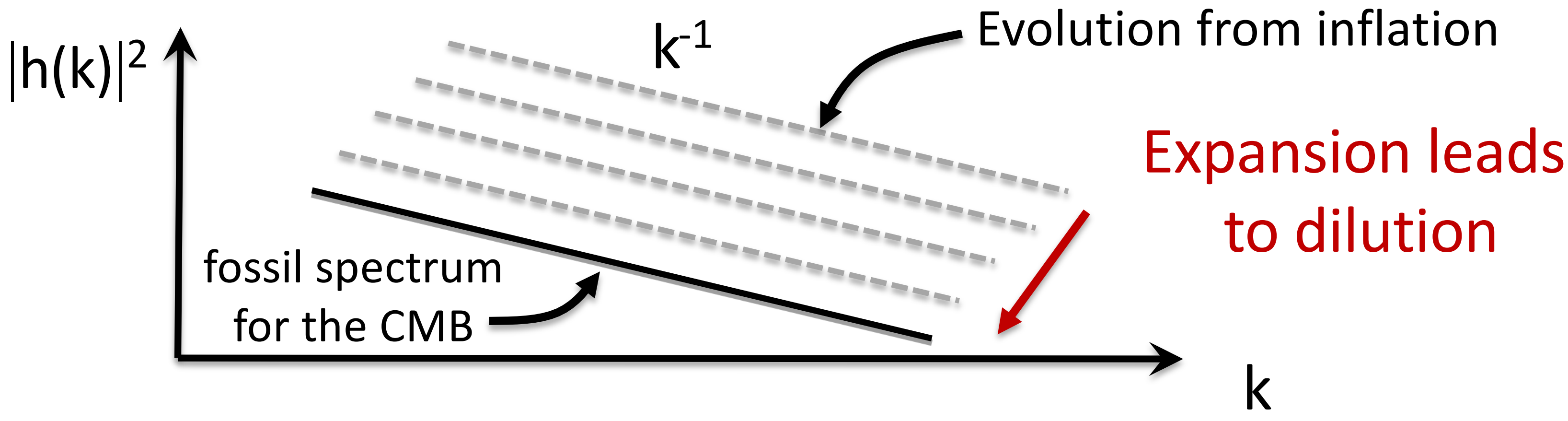
$$g_{\mu\nu} = \overbrace{h_0(t)\eta_{\mu\nu}} + \eta_{\mu\nu} + h_{\mu\nu} = h_0(t)(\eta_{\mu\nu} + \tilde{h}_{\mu\nu})$$



The growth of $h_0(t)$ is interpreted as an expansion of the Universe

Inflation appears if the growth in time is fast enough

Comparison with observations

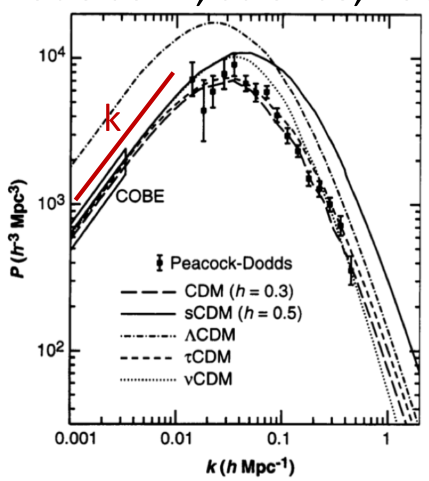


Small fluctuations are treated in the Newtonian limit: $\nabla^2\phi = 4\pi G\rho$

$$E_\ell \sim \frac{c^4}{32\pi G}\frac{h_\ell^2}{\ell^2} \Rightarrow P_\phi(k) = \phi^2(k) \sim k^{n_s-2} \sim k^{-1}$$

Prediction compatible with the Harrison-Zeldovich spectrum (n_s=1)
[Harrison, PRD, 1970; Zeldovich, MNRAS, 1972]

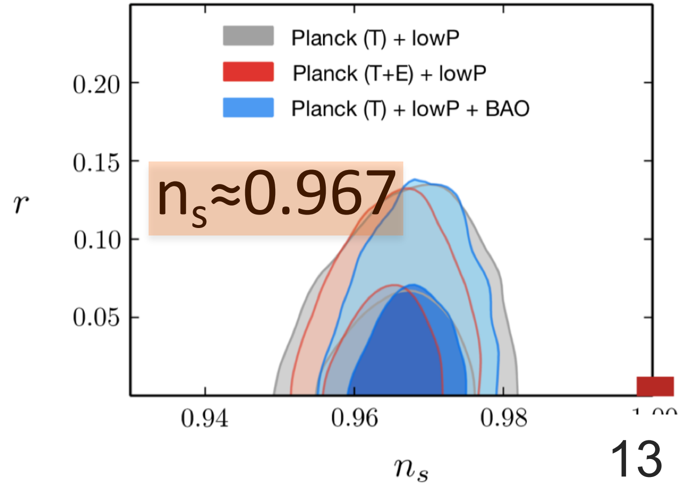
$P_\rho(\mathbf{k}) = \rho^2(\mathbf{k}) \sim \rho^2(k)/k^2 \sim k^{n_s} \sim k$
[Dodelson+, science, 1996]



($\delta T/T \sim \delta\rho/\rho \sim 10^{-5}$)

Planck is compatible with the fossil spectrum $|h(k)|^2 \sim k^{-1.033}$

Latest data
[Planck coll., A&A, 2015]



Conclusion

- Theory of weak GW – space-time – turbulence (4 waves)
- Explosive inverse cascade of wave action / **anomalous** scaling
- The Riemann (4th order) curvature tensor is **non-trivial**

- Phenomenological (CB) model of inflation (via a condensate)
  - ➔ a **standard** model of inflation ('no' tuning parameter)
  - ➔ **falsifiable** predictions (with dns of general relativity)
  - ➔ close to elastic wave turbulence [Hassaini+, 2018]
- **Fossil spectrum ~ compatible with CMB data**
- Presence of an **anomalous** scaling in the Planck data (n_s<1)
  - ➔ necessary to explain the structures in the Universe
  - ➔ can this anomalous scaling be explained by turbulence ?

[Semikoz & Tkachev, PRL, 1995 ; Lacaze+, PD, 2001]