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On the Zakharov-L'vov stochastic model for wave turbulence

based on a joint work with Andrey Dymov (Moscow)

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§1. The setting

Consider the modified NLS equation

$$\frac{\partial}{\partial t}u + i\Delta u - i\nu\left(|u|^2 - \|u\|^2\right)u = 0\,,$$

$$\Delta = (2\pi)^{-2}\sum_{j=1}^{d}(\partial^2/\partial x_j^2)\,,\quad x\in\mathbb{T}_L^d=\mathbb{R}^d/(L\mathbb{Z}^d)\,,$$

where $d\geq 2$, $L\geq 1$ and $\nu\in(0,1]$. This is a hamiltonian PDE, obtained by modifying the standard NLS equation by another hamiltonian equation $\frac{\partial}{\partial t}u=-i\nu\|u\|^2u$, whose flow commutes with that of NLS. This is a rather innocent modification.

Denote by H the space $L_2(\mathbb{T}_L^d;\mathbb{C})$, given the normalised L_2–norm

$$\|u\|^2 = L^{-d}\Re\int|u|^2\,dx\,;\qquad \text{so}\ \ \|\mathbf{1}\|=1.$$

We write solutions u as $u(t,x)$ or as $u(t) \in H$. Pass to the slow time $\tau = \nu t$:

$$\dot{u} + i\nu^{-1}\Delta u - i\left(|u|^2 - \|u\|^2\right)u = 0\,, \quad \dot{u} = (\partial/\partial\tau)u(\tau,x), \ \ x \in \mathbb{T}^d_L\,.$$

From now on I will use the time τ.

The objective is to study solutions when $\nu \to 0$ and $L \to \infty$.

We write the Fourier series for $u(x)$ as

$$u(x) = L^{-d/2}\sum\nolimits_{s\in\mathbb{Z}^d_L} v_s e^{2\pi i s\cdot x}, \qquad \mathbb{Z}^d_L = L^{-1}\mathbb{Z}^d\,,$$

where $v_s = L^{-d/2}\int_{\mathbb{T}^d_L} u(x)e^{-2\pi i s\cdot x}\,dx\,.$

When studying the equation, people talk about “pumping the energy to low modes and dissipating it in high modes”. To make this rigorous, Zakharov-L’vov in 1975 suggested to consider the NLS equation, dumped by a (hyper)viscosity and driven by a random force:

$$\begin{aligned} &\dot{u} + i\nu^{-1}\Delta u - i\rho\,(|u|^2 - \|u\|^2)u = -(-\Delta+1)^{r_*}u + \dot{\eta}^{\omega}(\tau,x), \\ &\eta^{\omega}(\tau,x) = L^{-d/2}\sum\nolimits_s b_s\beta_s^{\omega}(\tau)e^{2\pi i s\cdot x}. \end{aligned} \tag{1}$$

Here $r_* > 0$, $\rho \geq 1$ is an additional constant, needed later, $\{\beta_s(\tau), s \in \mathbb{Z}_L^d\}$ are standard independent complex Wiener processes, the constants $b_s > 0$ are defined for all $s \in \mathbb{R}^d$ and fast decay when $|s| \to \infty$.

Denoting $B = L^{-d}\sum_s b_s^2$ we obtain the balance of energy for solutions of (1):

$$\mathbb{E}\|u(\tau)\|^2 + 2\mathbb{E}\int_0^\tau \|(-\Delta+1)^{r_*}u(s)\|^2\,ds = \mathbb{E}\|u(0)\|^2 + 2B\tau\,.$$

So the quantity $\mathbb{E}\|u(\tau)\|^2$ – the averaged “energy per volume” of a solution u – is order one, uniformly in L, how this should be.

Passing to the Fourier presentation, we write eq. (1) as

$$\dot v_s - i\nu^{-1}|s|^2 v_s + \gamma_s v_s = i\rho L^{-d}\sum\nolimits_{1,2}\delta'^{12}_{3s} v_1 v_2 \bar v_3 + b_s\dot\beta_s\,,\qquad s\in\mathbb{Z}^d_L,$$

where $\gamma_s = (1+|s|^2)^{r_*}$ and

$$\delta'^{12}_{3s} = \begin{cases} 1, & \text{if } s_1+s_2 = s_3+s \text{ and } \{s_1,s_2\}\neq\{s_3,s\}, \\ 0, & \text{otherwise.}\end{cases}$$

Using interaction representation $v_s = \exp(i\nu^{-1}\tau|s|^2)\, a_s$ we write equations for v_s as

$$\dot a_s + \gamma_s a_s = i\rho \mathcal{Y}_s(a;\nu^{-1}\tau) + b_s\dot\beta_s\,, \quad s\in\mathbb{Z}^d_L\,,$$

(2) $$\mathcal{Y}_s(a;t) = L^{-d}\sum\nolimits_{1,2}\delta'^{12}_{3s} a_1 a_2 \bar a_3 e^{it\omega^{12}_{3s}},$$

$$\omega^{12}_{3s} = |s_1|^2+|s_2|^2-|s_3|^2-|s|^2 = -2(s_1-s)\cdot(s_2-s).$$

The *energy spectrum* of a solution $u(\tau)$ is the function

$$\mathbb{Z}^d_L \ni s\mapsto n_s(\tau) = n_s^{L,\nu}(\tau) = \mathbb{E}|v_s(\tau)|^2 = \mathbb{E}|a_s(\tau)|^2.$$

Traditionally the function n_s is in the center of attention. We wish to study the solutions of (1) and their energy spectra n_s when

$$\nu\to 0,\quad L\to\infty.$$

Exact meaning of this assumption is not clear. Below we specify it as follows:

$\nu\to 0$ and $L\geq \nu^{-2-\epsilon}$ for some $\epsilon>0$,

or first $L\to\infty$ and next $\nu\to 0$.

§2. Solutions as formal series in ρ.

Consider the equations with the initial condition

$$u(-T) = 0, \qquad 0 < T \leq +\infty,$$

and write the solution a_s as formal series in ρ:

$$a_s = a_s^{(0)} + \rho a_s^{(1)} + \dots.$$

Substituting this decomposition in the a–equation (2), we see that

$$\dot{a}_s^{(0)}(\tau) + \gamma_s a_s^{(0)}(\tau) = b_s \beta_s(\tau), \quad s \in \mathbb{Z}_L^d.$$

So the processes $a_s^{(0)}$ are independent Ornstein–Uhlenbeck processes:

$$a_s^{(0)}(\tau) = b_s \int_{-T}^{\tau} e^{-\gamma_s(\tau - l)} d\beta_s(l),$$

while $a^{(1)}$ satisfies

$$\dot{a}_s^{(1)}(\tau) + \gamma_s a_s^{(1)}(\tau) = i\mathcal{Y}_s(a^{(0)}(\tau); \nu^{-1}\tau), \qquad \tau > -T\,,$$

so

$$a_s^{(1)}(\tau) = i\int_{-T}^{\tau} e^{-\gamma_s(\tau-l)}\mathcal{Y}_s(a^0(l); \nu^{-1}l)\,dl.$$

That is, $a_s^{(1)}(\tau)$ is a Wiener chaos of third order. Similar, for $n \geq 2$,

$$a_s^{(n)}(\tau) = i\int_{-T}^{\tau} \sum_{n_1+n_2+n_3=n-1} e^{-\gamma_s(\tau-l)}\mathcal{Y}_s(a_1^{(n_1)}(l), a_2^{(n_2)}(l), a_3{}^{(n_3)}(l); \nu^{-1}l\big)\,dl\,,$$

is a Wiener chaos of order $2n+1$.

QUASISOLUTIONS. The NLS equation is a model which is used to describe various small–amplitude nonlinear processes, neglecting the terms, cubic in the amplitude. So what has real physical meaning rather is not itself a solution $a_s(\tau)$ of the a–equation (2), but its quadratic in ρ part. In the notation above this is :

$$A_s(\tau) = a_s^0(\tau) + \rho a_s^1(\tau) + \rho^2 a_s^2(\tau).$$

We call the the process $A = \{A_s(\tau)\}$ the QUASISOLUTION.

Consider the energy spectrum of A,

$$N_s(\tau) = \mathbb{E}|A_s(\tau)|^2.$$

QUESTION: How N_s behaves when $\nu \to 0$ and $L \to \infty$, $L \gg \nu^{-2}$?

Let us write N_s as series in ρ:

$$N_s(\tau) = n_s^0(\tau) + \rho n_s^1(\tau) + \rho^2 n_s^2(\tau) + \rho^3 n_s^3(\tau) + \rho^4 n_s^4(\tau).$$

Here $n_s^0 = \mathbb{E}|a_s^0|^2$ is a quantity of order 1,

$$n_s^1 = 2\Re\mathbb{E} a_s^0 \bar{a}_s^1 = 0, \quad n_s^2 = \mathbb{E}|a_s^1|^2 + 2\Re\mathbb{E} a_s^0 \bar{a}_s^2, \text{ etc.}$$

CALCULATION: if $\nu \ll 1$ and $L \gg \nu^{-2}$, then

$$n_s^2 \sim \nu, \qquad n_s^3, n_s^4 \lesssim \nu^2.$$

So the right scaling for ρ is $\rho \sim \nu^{-1/2}$. Accordingly let us take ρ in the form

$$\rho = \sqrt{\varepsilon}\, \nu^{-1/2}, \quad \varepsilon \in (0, 1].$$

§3. Wave kinetic equation

For a real function $s \mapsto x_s$ on $\mathbb{R}^d$ let us consider the Cubic Wave Kinetic Integral

$$K_s(x.) = 2\pi\gamma_s \int_{\Sigma_s} \frac{ds_1\, ds_2\,|_{\Sigma_s}\;\; x_1 x_2 x_3 x_s}{\sqrt{|s_1 - s|^2 + |s_2 - s|^2}} \left(\frac{1}{\gamma_s x_s} + \frac{1}{\gamma_3 x_3} - \frac{1}{\gamma_1 x_1} - \frac{1}{\gamma_2 x_2} \right).$$

Here $x_j = x_{s_j}$, $j = 1, 2, 3$, we substitute $s_3 = s_1 + s_2 - s$,

$$\Sigma_s = \{(s_1, s_2) : (s_1 - s) \cdot (s_2 - s) = 0\},$$

and $ds_1\, ds_2\,|_{\Sigma_s}$ is the microcanonical measure on Σ (the volume in $\mathbb{R}^{2d}$, restricted to Σ).

FACT: the Wave Kinetic operator $x_s \to K_s(x.)$ is well defined and “good”.

Consider the Wave Kinetic Equation:

$$(WKE) \qquad \dot{m}_s(\tau) = -2\gamma_s m_s(\tau) + 2b_s^2 + \varepsilon K_s(m.(\tau)), \quad s \in \mathbb{R}^d.$$

For small ε this is a good equation. It has a unique solution, equal 0 at $-T$. Let us denote it $\{n_s^*(\tau)\}$.

Theorem. Let $\rho = \sqrt{\varepsilon}\, \nu^{-1/2}$, where ε is a small constant. Then the energy spectrum $N_s(\tau)$ is close to the solution $n_s^*(\tau)$ of (WKE):

$$\|n_s^*(\tau) - N_s(\tau)\| \leq C\varepsilon^2 \qquad \forall\, \tau \geq -T.$$

The solution $n_s^*(\tau)$ can be written as

$$n_s^*(\tau) = n_s^{*0}(\tau) + \varepsilon n_s^{*1}(\tau) + O(\varepsilon^2),$$

where $n_s^{*0}(\tau) \sim 1$ solves the linear equation $(WKE)_{\varepsilon=0}$, and $\varepsilon n_s^{*1}(\tau)$ is the nonlinear part of the solution.

Note that $C\varepsilon^2 \ll |\varepsilon n_s^{*1}(\tau)|$ for small ε.

Remark. If $\rho = \sqrt{\varepsilon}\,\nu^{-1/2}$, then in the original fast time t the equation reeds:

$$\frac{\partial}{\partial t}u+i\Delta u-i\sqrt{\nu}\,\sqrt{\varepsilon}\,\big(|u|^2-\|u\|^2\big)u = -\nu(-\Delta+1)^{r_*}u+\sqrt{\nu}\,\dot\eta^{\omega}(\tau,x), \quad \|u(t)\|\sim 1.$$

That is,

1) the time-scale which we use to pass to the kinetic limit is $\tau = \nu t$, so the time needed to arrive at the limiting kinetic regime is $t\sim\nu^{-1}$;

2) the coefficient in front of the nonlinearity is

$$\nu^{1/2}.$$

§4. Higher order in ρ decompositions

Write the solution in a–presentation as formal series in ρ:

$$a = a^{(0)} + \rho a^{(1)} + \dots,$$

and accordingly write its energy spectrum as

$$n_s(\tau) = n_s^0(\tau) + \rho n_s^1(\tau) + \rho^2 n_s^2(\tau) + \dots . \qquad (3)$$

Since

$$n_s^2 \sim \nu, \qquad n_s^3, n_s^4 \lesssim \nu^2,$$

it is natural to assume that

$$n_s^k \lesssim \nu^{k/2} \quad \text{for all } \nu \text{ and all } k.$$

If so, then scaling as before $\rho = \sqrt{\varepsilon}\,\nu^{-1/2}$, we would make (3) a nice asymptotical series in ε.

But this is WRONG:

Theorem. 1) For each k we have

$$n_s^k \leq C_s^{\#}(k) \max(\nu^{\lceil k/2 \rceil}, \nu^d),$$

where $\lceil k/2 \rceil$ – the smallest integer which is $\geq k/2$.
2) Moreover, if $k > 2d$, then the sum of the integrals which makes the term n_s^k contains integrals of order $\sim \nu^d \gg \nu^{\lceil k/2 \rceil}$.

The integrals of order ν^d do not cancel each other. So for big k

$$n_s^k \sim \nu^d, \quad \text{NOT} \quad n_s^k \sim \nu^{\lceil k/2 \rceil}.$$

Then the series

$$n_s(\tau) = n_s^0(\tau) + \rho n_s^1(\tau) + \rho^2 n_s^2(\tau) + \dots .$$

with the right scaling $\rho = \sqrt{\varepsilon}\, \nu^{-1/2}$ IS NOT an asymptotical series since

$$\rho^k n_s^k(\tau) > \varepsilon^{k/2} \nu^{d-k/2},$$

which is very big for $k > 2d$ and $\nu \ll 1$.