SYMMETRY BREAKINGS IN THE WAKE OF A DISK: A GLOBAL STABILITY ANALYSIS

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Summary The onset of unsteadiness in the wake of a circular flat disk is investigated in the framework of the global stability theory. We address the connection between the bifurcations undergone respectively by the axisymmetric steady state and by the real flow. A model based on the normal form theory is presented, that allows to identify three successive bifurcations. The critical Reynolds numbers and the time and space symmetries of the stable solutions show excellent agreement with the results of direct numerical simulations.

INTRODUCTION

The wake past an axisymmetric disk has already been investigated using the framework of linear global stability [4]. This study predicted a first bifurcation for Re=116.5 leading to the loss of axisymmetry, and a second bifurcation for Re=125.6 leading to unsteadiness (loss of time invariance). However, these values were obtained by considering the stability of a steady axisymmetric base flow, which no longer exists for Re>116.5. In this paper, we use the normal form theory to clarify the connection between the bifurcations undergone respectively by the axisymmetric steady state and the real flow. A similar study has been carried out by use of direct numerical simulations [2]. We address the question using the global stability theory to build a model retaining the lowest-order nonlinear terms that respect the symmetries of the initial problem. The symmetry properties and the dynamics of the model are then compared to that of the whole system.

METHODOLOGY AND RESULTS

We consider a disk of diameter D in a uniform flow of velocity U_∞. Standard cylindrical coordinates r, θ and z with origin taken at the center of the disk are used. The fluid motion is governed by the incompressible Navier-Stokes equations made non-dimensional by D and U_∞. $\mathbf{u} = (u_r, u_\theta, u_z)$ is the fluid velocity where u_r, u_θ and u_z are the radial, azimuthal and axial components, and p is the pressure. We consider a flow $\mathbf{q} = (\mathbf{u}, p)$ made of the superposition of an axisymmetric steady base flow $\mathbf{q_0} = (u_{r0}, 0, u_{z0}, p_0)$ and a three-dimensional perturbation $\mathbf{q_1} = \epsilon(u_{r1}, u_{\theta 1}, u_{z1}, p_1)$ of amplitude ϵ.

The base flow is obtained through time-dependent simulations, based on second order schemes (Taylor-Hood finite-elements and Lagrange-Galerkin discretizations). The instability problem for $\mathbf{q_1}$ is formulated simultaneously linearizing the governing equations about $\mathbf{q_0}$. We use a global normal mode expansion

$$\mathbf{q_1} = \hat{\mathbf{q}}_1(r,z)e^{\sigma t + im\theta} + \text{ c.c.} \tag{1}$$

where $\hat{\mathbf{q}}_1 = (\hat{u}_{r1}, \hat{u}_{\theta 1}, \hat{u}_{z1}, \hat{p}_1)$ is the so-called global mode. m is the integer azimuthal wavenumber and σ is the complex pulsation, σ_r and σ_i being respectively the growth rate and frequency of the global mode. $\hat{\mathbf{q}}_1$ and σ are solutions of an eigenvalue problem which is then solved numerically (Arnoldi method) to evaluate the critical Reynolds numbers.
Results of our global stability analysis are consistent with that of ([4]). The first bifurcation occurs at $\text{Re}_{c1} = 117.0$ for an $m = 1$ non-oscillating global mode $\hat{\mathbf{q}}_1^{\text{A}}$ ($\sigma_i = 0$). The spatial structure of the associated eigenmode displays strong large-scale axial velocity disturbances under the form of a pair of counter-rotating streamwise vortices responsible for the loss of axisymmetry (figure 1). A Hopf bifurcation then occurs at $\text{Re}_{c2} = 125.4$ for both an $m = +1$ global mode $\hat{\mathbf{q}}_1^{\text{B+}}$ and an $m = -1$ global mode $\hat{\mathbf{q}}_1^{\text{B-}}$ of same frequency $\sigma_0 = 0.760$ ($St = fD/U_\infty = 0.12$), whose associated eigenmode exhibits a spatially periodic downstream structure characteristic of the oscillatory wake instability (figure 2).

We consider the model flow consisting of the superposition of $\mathbf{q_0}$ and of the three modes $\hat{\mathbf{q}}_1^{\text{A}}$, $\hat{\mathbf{q}}_1^{\text{B+}}$ and $\hat{\mathbf{q}}_1^{\text{B-}}$, undergoing a multiple codimension bifurcation at the critical Reynolds number $\text{Re}_{c2} = 125.4$. We introduce three time-dependent complex amplitudes A, B^+ and B^- so that

$$\hat{\mathbf{q}}_1 = A\hat{\mathbf{q}}_1^{\text{A}}e^{i\theta} + B^+\hat{\mathbf{q}}_1^{\text{B+}}e^{i\theta + i\sigma_0 t} + B^-\hat{\mathbf{q}}_1^{\text{B-}}e^{-i\theta + i\sigma_0 t} + \text{ c.c.} \tag{2}$$

A system of coupled Stuart-Landau amplitude equations retaining only the lowest-order nonlinear terms is derived by considering that invariance is required when the origins of time ($t \to t + t_0$) and azimuthal positions ($\theta \to \theta + \theta_0$) are changed, and when the $\theta \to -\theta$ symmetry is applied. The system of amplitude equations finally reads

$$dA/dt = \epsilon^2 \Delta Re^{-1}\lambda_{\text{A}} A - \epsilon^2 A(\mu_{\text{A}}|A|^2 + \nu_{\text{A}}|B^+|^2 + \overline{\nu}_{\text{A}}|B^-|^2) - \epsilon^2\chi_{\text{A}} B^+\overline{B^- A} \tag{3a}$$

$$dB^+/dt = \epsilon^2 \Delta Re^{-1}\lambda_{\text{B}} B^+ - \epsilon^2 B^+(\mu_{\text{B}}|B^+|^2 + \nu_{\text{B}}|B^-|^2 + \eta_{\text{B}}|A|^2) - \epsilon^2\chi_{\text{B}} B^- A^2 \tag{3b}$$

$$dB^-/dt = \epsilon^2 \Delta Re^{-1}\lambda_{\text{B}} B^- - \epsilon^2 B^-(\mu_{\text{B}}|B^-|^2 + \nu_{\text{B}}|B^+|^2 + \eta_{\text{B}}|A|^2) - \epsilon^2\chi_{\text{B}} B^+\overline{A}^2 . \tag{3c}$$

All coefficients are numerically computed using the adjoint global modes $\hat{\mathbf{q}}_{1\text{A}}^\dagger$, $\hat{\mathbf{q}}_1^{\text{B+}\dagger}$ and $\hat{\mathbf{q}}_1^{\text{B+}\dagger}$, each adjoint mode being solution of a specific eigenvalue problem. The interpretation of these adjoint modes will be discussed in terms of receptivity to external forcing. This point is of particular importance when considering experimental set-ups, for instance the disk holding device induces perturbations that may be seen as local modifications of the base flow. We obtain

$$
\begin{array}{l|l}
\lambda_{\mathrm{A}} = 65 & \lambda_{\mathrm{B}} = 67 + 13\,\mathrm{i} \\
\mu_{\mathrm{A}} = 2.9 & \mu_{\mathrm{B}} = 0.33 - 0.015\,\mathrm{i} \\
\nu_{\mathrm{A}} = 0.89 - 0.20\,\mathrm{i} & \nu_{\mathrm{B}} = 0.44 - 0.15\,\mathrm{i} \\
 & \eta_{\mathrm{B}} = 0.94 - 3.2\,\mathrm{i} \\
\chi_{\mathrm{A}} = 0.69 & \chi_{\mathrm{B}} = 1.4 - 1.1\,\mathrm{i}\,.
\end{array}
$$

A mathematical exploration of the solutions of system (3) is available in [3]. These solutions show excellent agreement with the results of time integrations (Runge-Kutta, 4th order), which allows us to build a consistent bifurcation diagram. (figure 3). The model flow undergoes a first bifurcation for Re=117.0: the axisymmetry is lost but the time invariance is preserved, leading to a 3D steady state with a reflectional symmetry. A Hopf bifurcation then occurs for Re=123.6, where both the remaining reflectional symmetry and the time invariance are broken, leading to a fully 3D periodic state. It should be pointed out that the loss of time invariance occurs slightly earlier than predicted by the linear stability analysis (Re=125.4), due to the fact that the steady axisymmetric base flow no longer exists in this range of Reynolds numbers. A third bifurcation then occurs for Re=139.6, where the flow remains unsteady, but recovers a lost reflectional symmetry. These results are consistent with direct numerical simulations [1,2] predicting three successive bifurcations at Reynolds numbers Re=115.5, 121.5 and 139.5, with identical symmetry features. This suggests that the 3D dynamics of the whole system is efficiently captured using a reduced order model based on the destabilization of the axisymmetric steady state.

References

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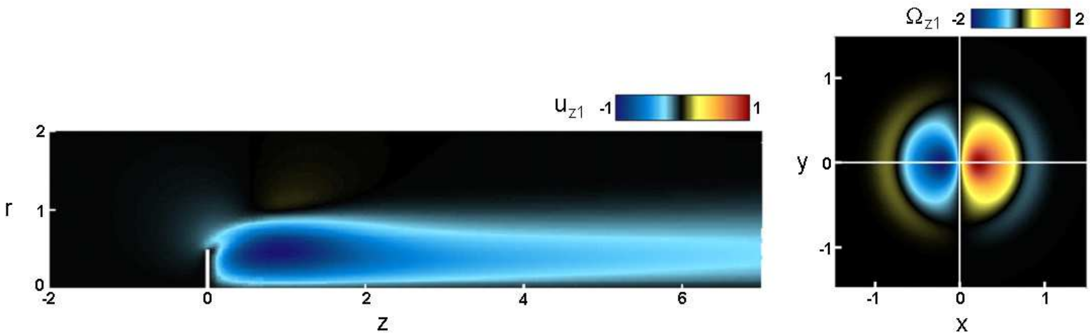

Figure 1. Axial velocity $\hat{u}^{\mathrm{A}}_{z1}$ and axial vorticity $\hat{\Omega}^{\mathrm{A}}_{z1}$ at $z = 2$ for the steady global mode (Re $= 117.0$, arbitrary normalization).

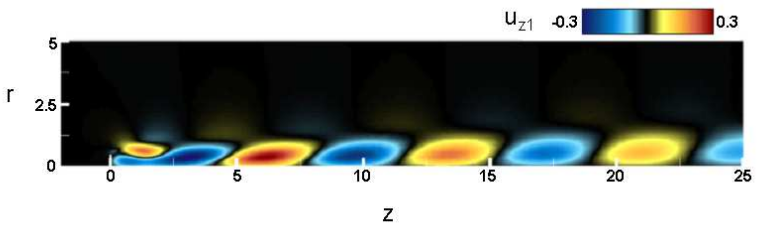

Figure 2. Axial velocity $\hat{u}^{\mathrm{B}+}_{z1}$ for the oscillating global mode (Re $= 125.4$, arbitrary normalization).

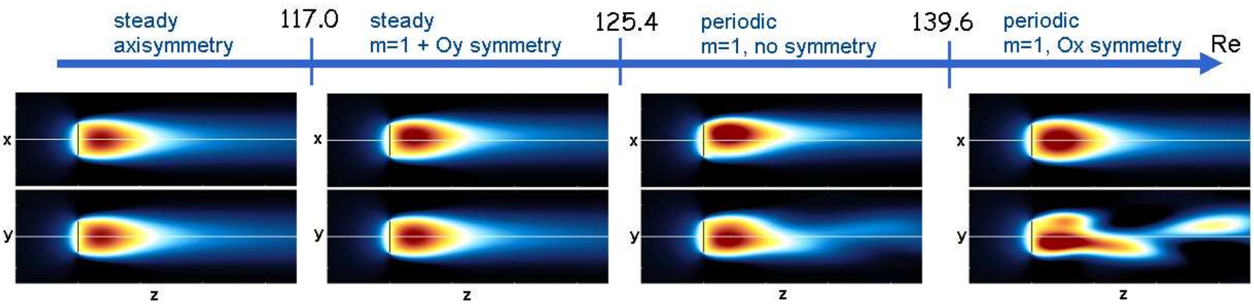

Figure 3. Simplified bifurcation diagram and related stable solutions.