Low-Reynolds number vortex-induced vibrations in the wake of a circular cylinder by means of adjoint methods

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Introduction

A large body of works has been devoted to the problem of the wake flow past a fixed cylinder, and it is now well known that at a critical Reynolds number $Re_* \sim 47$, the flow undergoes a global instability responsible for the onset of the vortex-shedding phenomenon, thus leading from a steady symmetric state towards a time-periodic non-symmetric state [1]. This causes the cylinder to experience unsteady lift and drag forces. If mounted on elastic supports, the cylinder may thus undergo vortex-induced vibrations (VIV). For a review of the recent progress achieved in this field, the reader is referred to Williamson & Govardhan [2]. The present work aims at investigating VIV in the vicinity of the critical Reynolds number Re_* by combining asymptotic analyses and adjoint-based receptivity methods. The main advantage of such an approach lies in the fact that it requires no specific treatment in the numerics, as for instance mesh deformation schemes [3].

1. Theoretical formalism

We consider a cylinder of diameter D in a uniform flow of velocity U_∞. Standard cylindrical coordinates x, y with origin taken at the center of the cylinder are used. The fluid motion is governed by the incompressible Navier-Stokes equations made non-dimensional by D and U_∞. $\mathbf{u} = (u, v)^T$ is the fluid velocity, with u and v the streamwise and transverse components, and p is the pressure. The state vector $\boldsymbol{q} = (\boldsymbol{u}, p)^T$ obeys the incompressible Navier–Stokes equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0\,, \qquad \partial_t \boldsymbol{u} + \boldsymbol{\nabla} \boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{\nabla} p - \frac{1}{Re} \boldsymbol{\nabla}^2 \boldsymbol{u} = 0\,, \tag{1}$$

written formally as

$$\mathcal{B} \partial_t \boldsymbol{q} + \mathcal{M}(\boldsymbol{q}, Re) = \boldsymbol{0}\,, \tag{2}$$

with Re the Reynolds number and $\mathcal{B}$ and $\mathcal{M}$ differential operators.

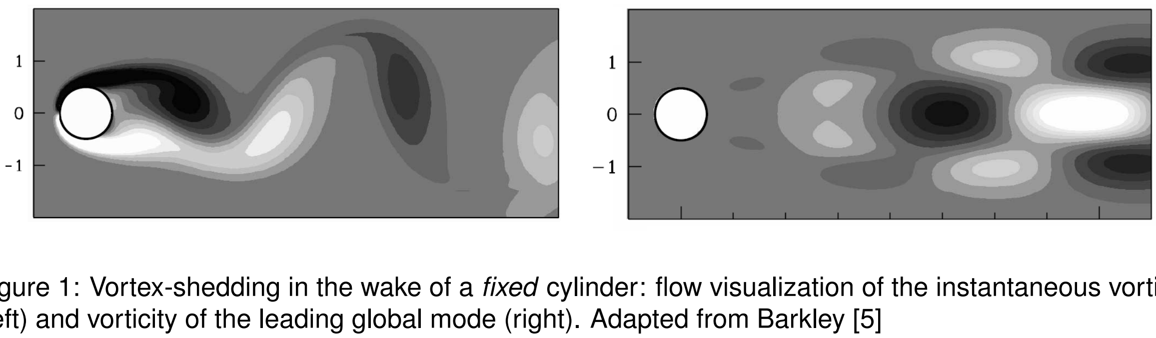

Figure 1: Vortex-shedding in the wake of a *fixed* cylinder: flow visualization of the instantaneous vorticity (left) and vorticity of the leading global mode (right). Adapted from Barkley [5]

The genesis of vortex-shedding past a *fixed* circular cylinder has been widely studied in the framework of the global stability theory, in which one characterizes the stability of the steady solution $\boldsymbol{q}_0$ to perturbations $\boldsymbol{q}_1$ of infinitesimal amplitude ϵ expanded as

$$\boldsymbol{q}_1(x, y, t) = \hat{\boldsymbol{q}}_1(x, y) e^{(\sigma + \mathrm{i}\omega)t} + \text{c.c.}\,. \tag{3}$$

In (3), σ and ω are the growth rate and pulsation of the global eigenmode $\hat{\boldsymbol{q}}_1$, and c.c. denotes the complex conjugate of the preceding expression. For Reynolds number $Re \geq Re_* \simeq 47$, the steady flow becomes unstable to a global mode of marginal pulsation $\omega_* \simeq 0.74$. The associated eigenvector $\hat{\boldsymbol{q}}_{1A}$ is solution of the generalized eigenvalue problem

$$(\mathrm{i}\omega_*\mathcal{B} + \mathcal{A}_*)\hat{\boldsymbol{q}}_{1A} = \boldsymbol{0}\,, \tag{4}$$

where $\mathcal{A}_*$ is the Navier–Stokes operator linearized around $\boldsymbol{q}_0$ at the critical Reynolds number [4,5,6].

In the present study, the cylinder is assumed to be mounted on springs in the transverse direction. In the following, y_s, $\dot{y}_s$ and $\ddot{y}_s$ denote the non-dimensional transverse displacement, velocity and acceleration of the cylinder center, whose motion is governed by the linear oscillator equation

$$\ddot{y}_s + 2\omega_s\gamma\dot{y}_s + \omega_s^2 y_s = \frac{2}{\pi m} C_y\,. \tag{5}$$

In (5), ω_s is the dimensionless oscillation frequency, γ is the structural damping coefficient and m is the mass number defined as the ratio of the solid to the fluid density $m = \rho_s/\rho$. C_y is the instantaneous lift coefficient of the cylinder per unit length, defined as

$$C_y = 2\int_{\Upsilon_w}\left(-p\boldsymbol{n} + \frac{1}{Re}\frac{\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T}{2}\cdot\boldsymbol{n}\right)\cdot\boldsymbol{e}_y \mathrm{d}l\,, \tag{6}$$

where $\boldsymbol{n}$ is the outward-pointing normal vector to the cylinder and $\mathrm{d}l$ the length element along the cylinder wall Υ_w. The coupling between the fluid and structural motions occurs through the impermeability condition imposed at the cylinder wall:

$$\boldsymbol{u}(x, y + y_s) = (0, \dot{y}_s)^T\,, \quad \forall (x,y) \in \Upsilon_w\,. \tag{7}$$

2. Asymptotic expansion

We carry out here an asymptotic expansion of the coupled flow-structure system, assuming that the Reynolds number Re departs from criticality at order ϵ^2. To this end, we introduce multiple time scales with a *fast* time scale t and a *slow* time scale $T = \epsilon^2 t$. We define the order unity parameter δ, such that

$$\frac{1}{Re} = \frac{1}{Re_*} - \epsilon^2\delta\,, \tag{8}$$

and expand the flow field $\boldsymbol{q}$ as

$$\boldsymbol{q} = \boldsymbol{q}_0 + \epsilon\boldsymbol{q}_1 + \epsilon^2\boldsymbol{q}_2 + \epsilon^3\boldsymbol{q}_3 + \dots\,. \tag{9}$$

We assume that the flow is forced by a near-resonance, third-order cylinder motion. The displacement is therefore written as

$$y_s = \epsilon^3 Y(T)e^{\mathrm{i}\omega_s t} + \text{ c.c.}\,, \tag{10}$$

where the complex amplitude Y is at this stage an unknown function of the slow time $T = \epsilon^2 t$. Anticipating on the dominant balance, we use the scaling laws

$$\gamma = \epsilon^2\Gamma + O(\epsilon^3)\,, \qquad \omega_s = \omega_*(1 + \epsilon^2\Omega) + O(\epsilon^3)\,, \qquad \frac{1}{m} = \epsilon^4\frac{1}{M} + O(\epsilon^5)\,, \tag{11}$$

where Γ, Ω and M are order one parameters. Substitution of the preceding expansions into (7) yields the series of boundary conditions at the cylinder wall:

$$\boldsymbol{u_i} \;=\; \boldsymbol{0}\ \ (i = 0\dots 2)\,, \tag{12a}$$

$$\boldsymbol{u}_3 \;=\; \boldsymbol{U_w}(T)e^{\mathrm{i}\omega_* t} + \text{ c.c.}\,, \quad \text{with} \quad \boldsymbol{U_w} = (\mathrm{i}\omega_*\boldsymbol{I} - \boldsymbol{\nabla u}_0)\cdot\big(0, Y\big)^T\,. \tag{12b}$$

Owing to the present balance, one sees from (12) that the flow problem is identical to that of the *fixed* cylinder up to the second order ϵ^2. The flow motion is forced by the cylinder displacement at order ϵ^3 through an equivalent resonant blowing and suction stemming from the cylinder displacement. This means in particular that taking into account the coupling of the flow and structural motions does not require any specific treatment in the numerics.

2.1 Flow model

The equations at order ϵ^0 define $\boldsymbol{q}_0$ as the steady flow developing past a *fixed* cylinder, solution of the nonlinear equations

$$\mathcal{M}(\boldsymbol{q}_0, Re_*) = \boldsymbol{0}\,. \tag{13}$$

The equation at order ϵ are the linearized Navier–Stokes equations, that defines $\boldsymbol{q}_1$ as a superposition of global modes developing on $\boldsymbol{q}_0$. It can be chosen as the marginally stable mode $\hat{\boldsymbol{q}}_{1A}$ multiplied by some unknown complex amplitude A, depending on the slow time T, i.e.

$$\boldsymbol{q}_1 = A(T)\hat{\boldsymbol{q}}_{1A}e^{\mathrm{i}\omega_* t} + \text{ c.c.}\,. \tag{14}$$

At orders ϵ^2 and ϵ^3, we obtain inhomogeneous linear equations that can be understood as the harmonic linearized Navier-Stokes operator about $\boldsymbol{q}_0$ forced by terms involving quantities of lower orders. The homogeneous operator is non-degenerate at order ϵ^2 but degenerate at order ϵ^3, where the Fredholm alternative is used and compatibility conditions are applied. In the case of a *fixed* cylinder, this yields the classical Stuart-Landau amplitude equations for the complex amplitude A already derived by Sipp & Lebedev [6]. In the present case, the compatibility conditions must be modified so as to encompass the effect of the resonant boundary condition 12(b). This imposes the amplitudes A and Y to obey the relation

$$\frac{dA}{dT} = \lambda\delta A - \mu A|A|^2 + \alpha Y\,. \tag{15}$$

To compute the coefficients involved in equation (15), one must first compute the adjoint global mode $\hat{\boldsymbol{q}}^\dagger_{1A}$ solution of the adjoint eigenvalue problem

$$(-\mathrm{i}\omega_*\mathcal{B} + \mathcal{A}^\dagger_*)\hat{\boldsymbol{q}}^\dagger_{1A} = \boldsymbol{0}\,. \tag{16}$$

All coefficients arise then as analytical scalar products between the adjoint global mode and appropriate forcing terms of order ϵ^3. In particular, α is the complex coefficient defined by

$$\alpha = \mathcal{S}^{-1}\int_{\Upsilon_w}\left(\hat{p}^\dagger_{1A}\boldsymbol{n} + \frac{1}{Re_*}\boldsymbol{\nabla}\hat{\boldsymbol{u}}^\dagger_{1A}\cdot\boldsymbol{n}\right)\cdot(\mathrm{i}\omega_*\boldsymbol{e}_y - \boldsymbol{\nabla}\boldsymbol{u}_0\cdot\boldsymbol{e}_y)\,\mathrm{d}l\,, \tag{17}$$

with $\mathcal{S}$ the scalar product between the direct and global modes.

2.2 Structure model

With the present balance, the cylinder displacement is governed by the first-order equations

$$\frac{dY}{dT} = \omega_*(-\Gamma + \mathrm{i}\Omega)Y + \frac{\beta}{\omega_* M}A\,, \tag{18}$$

where β is the complex coefficient defined by

$$\beta = \frac{2\mathrm{i}}{\pi}\int_{\Upsilon_w}\left(\hat{p}_{1A}\boldsymbol{n} - \frac{1}{Re}\frac{\boldsymbol{\nabla}\hat{\boldsymbol{u}}_{1A} + \boldsymbol{\nabla}\hat{\boldsymbol{u}}_{1A}{}^T}{2}\cdot\boldsymbol{n}\right)\cdot\boldsymbol{e_y}\,\mathrm{d}l\,. \tag{19}$$

2.3 Coupled system

The coupled equations finally read

$$\begin{aligned}\frac{dA}{dT} &= \lambda\delta A - \mu A|A|^2 + \alpha Y + \eta\,, &\text{(20a)}\\ \frac{dY}{dT} &= \omega_*(-\Gamma + \mathrm{i}\Omega)Y + \frac{\beta}{\omega_* M}A\,. &\text{(20b)}\end{aligned}$$

To compute the parameter values, we set $\epsilon = 10^{-1}$, as a different choice would only yield a rescaling of the various coefficients. This yields:

$$\lambda = 9.15 + 3.24\mathrm{i}\,, \qquad \mu = 10.74 - 35.66\mathrm{i}\,, \qquad \alpha = 0.36 + 0.17\mathrm{i}\,, \qquad \beta = 0.17\,. \tag{21}$$

3. Limit-cycles and nonlinear dynamics

It possible to study the nonlinear dynamics of the coupled system (20) by setting $A = |A|e^{\mathrm{i}\psi_A}$ and $Y = |Y|e^{\mathrm{i}\psi_Y}$. To this end, we recast first (20) into a three-dimensional polar system for $|A|$, $|Y|$ and the phase $\phi = \psi_A - \psi_Y + \arg\beta$, which represents the phase shift between the forcing exerted by the cylinder motion and the vortex-shedding. By analyzing the dynamics of the associated limit cycles, it can be shown that the present adjoint-based model allows to recover the main phenomenology of vortex-induced vibrations: existence of a lock-in domain in which the vortex-shedding and the cylinder displacement synchronize, hysteretical behaviours (as shown in figure 2) and occurrence of vortex-shedding at subcritical Reynolds numbers $Re \leq Re_*$ [7,8]. The present formalism will ultimately be applied to the question of energy recovery: we will show that this model can indeed be used to optimize the amount of energy produced provided a suitable device is used to store the mechanical energy dissipated by the oscillating cylinder.

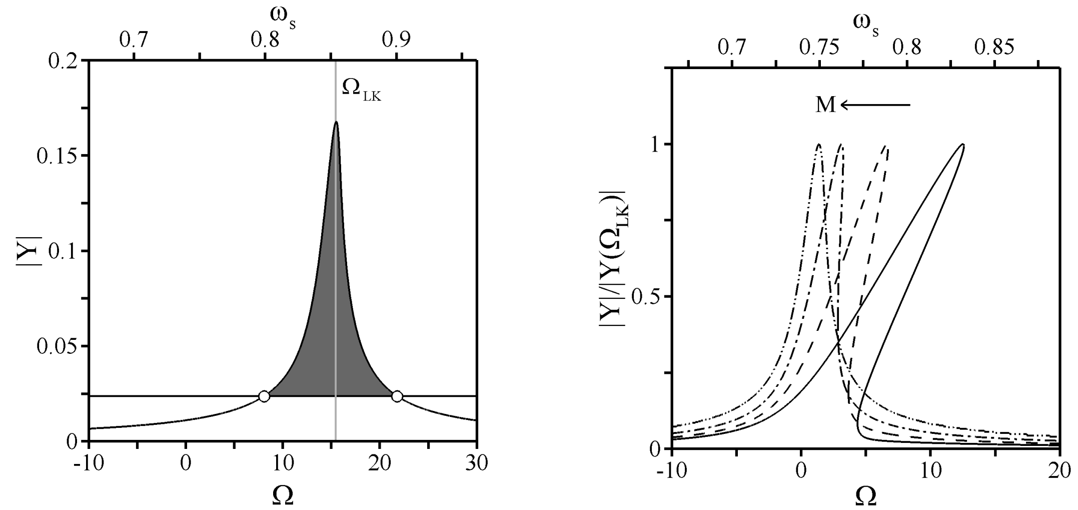

Figure 2: Left: amplitude of the cylinder displacement as a function of the detuning parameter ($Re = 55$, $\Gamma = 1$, $M = 1$). The dark grey shaded area corresponds to the resonance width for which $|Y|$ is larger than 2% of its maximal amplitude, hence evidencing the occurrence of lock-in. Right: same figure for various mass numbers ($Re = 47$, $\Gamma = 1$). For low mass ratios, a hysteretical behaviour occurs near the low and high ends of the lock-in regime.

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