Unsteadiness in the wake of the sphere: receptivity and weakly non-linear global stability analysis

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Introduction A large body of works has been devoted to the wake of the sphere in the last decades ([1,3,4,7,8]). For Reynolds numbers $\mathrm{Re} \gtrsim 280$, the flow is dominated by an instability of the helical mode, resulting in the low frequency shedding of large-scale coherent structures in the form of two superimposed modes of azimuthal wavenumbers $m = \pm 1$. Low Strouhal numbers of 0.2, characteristic of vortex shedding phenomena, have been reported, based on the body diameter. In this paper, we calculate the global modes leading the successive bifurcations undergone by the axisymmetric steady wake for $\mathrm{Re} < 300$. The corresponding adjoint global modes are computed, whose physical interpretation is discussed in terms of receptivity. These results are used to build an extended dynamical system for which we carry out a weakly non-linear stability analysis. A system of coupled Stuart-Landau amplitude equations is derived, aiming at giving a precise description of the periodic regime which appears after the transition from steady to unsteady wakes.

Base flow computation and global linear stability analysis We consider a sphere of diameter D in a uniform flow of velocity U_∞. Standard cylindrical coordinates r, θ and z with origin taken at the center of the sphere are used. The fluid motion is governed by the incompressible Navier-Stokes equations made non-dimensional by D and U_∞. $\mathbf{u} = (u, v, w)$ is the fluid velocity where u, v and w are the radial, azimuthal and axial components, and p is the pressure. The computational domain Ω is made of a single azimuthal plane. We impose standard boundary conditions on $\partial\Omega$, namely uniform inlet and no-strain outlet conditions, along with no-slip conditions on the sphere. The condition at the $r = 0$ axis depends on the solution symmetries and will be discussed further. The spatial discretization is achieved by use of Taylor-Hood finite-elements (P2 elements for $\mathbf{u}$ and P1 elements for p).

In the linear global stability theory, the aerodynamic flow field $\mathbf{q} = (\mathbf{u}, p)$ is decomposed into an axisymmetric steady base flow $\mathbf{q_0} = (u_0, 0, w_0, p_0)$ and a three-dimensional perturbation $\mathbf{q_1} = \epsilon^{1/2}(u_1, v_1, w_1, p_1)$ where $\epsilon^{1/2}$ is the small amplitude of the perturbation. The base flow is searched as a steady axisymmetric solution of the governing equations, verifying

$$\nabla.\mathbf{u_0} = 0 \qquad \nabla\mathbf{u_0}.\mathbf{u_0} + \nabla p_0 - \mathrm{Re}^{-1}\nabla^2\mathbf{u_0} = 0\,. \tag{1}$$

$\mathbf{q_0}$ is obtained from time-dependent simulations based on a Lagrange-Galerkin temporal discretization. Figure 1 shows the base flow obtained for a subcritical Reynolds number $\mathrm{Re} = 200$.

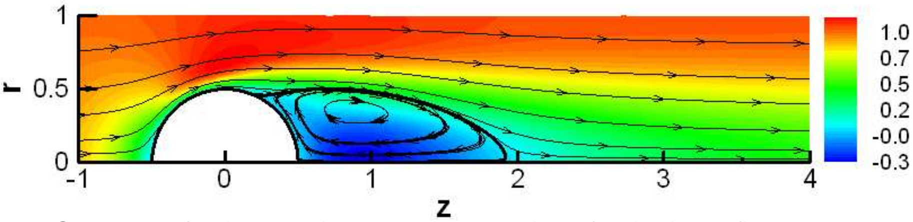

Figure 1: Contours of axial velocity w_0 and streamlines for the base flow $\mathbf{q_0}$ at $\mathrm{Re} = 200$.

At leading order in $\epsilon^{1/2}$, $\mathbf{q_1}$ is a solution of the unsteady equations linearized about $\mathbf{q_0}$

$$\nabla.\mathbf{u_1} = 0 \qquad \partial_t\mathbf{u_1} + C[\mathbf{u_0}, \mathbf{u_1}] + \nabla p_1 - \mathrm{Re}^{-1}\nabla^2\mathbf{u_1} = 0 \tag{2}$$

where $C[\mathbf{u}, \mathbf{v}]$ is the linearized convection operator $\nabla\mathbf{u}.\mathbf{v} + \nabla\mathbf{v}.\mathbf{u}$. Since all quantities are 2π periodic in the azimuthal direction, all perturbations are chosen in the form of global normal modes

$$\mathbf{q_1} = \hat{\mathbf{q}}_1(r, z)e^{\sigma t + im\theta} + \text{ c.c.} \tag{3}$$

where $\hat{\mathbf{q}}_1 = (\hat{u}_1, \hat{v}_1, \hat{w}_1, \hat{p}_1)$ is the so-called global mode. m is the integer azimuthal wavenumber and σ is the complex pulsation, σ_r and σ_i being respectively the growth rate and frequency of the global mode ($\sigma_r > 0$ for a global mode whose amplitude grows exponentially in time). Substitution of the development (3) in equations (2) leads to a generalized eigenvalue problem for σ and $\hat{\mathbf{q}}_1$ that reads

$$\mathcal{M}.\hat{\mathbf{q}}_1 = \sigma \mathcal{N}.\hat{\mathbf{q}}_1 \tag{4}$$

where $\mathcal{M}$ and $\mathcal{N}$ are two real matrices and $\hat{\mathbf{q}}_1$ is the complex eigenvector associated to σ. This eigenvalue problem is solved by use of an Arnoldi method based on a shift-invert strategy. The boundary conditions at the symmetry axis are derived from the asymptotic behavior of $\hat{\mathbf{q}}$ near the axis. We impose $u_0 = \partial_r w_0 = 0$ for the base flow and $\partial_r u_1 = \partial_r v_1 = w_1 = 0$ for $m = 1$ disturbances.

Results of the global stability analysis are consistent with that obtained by use of spectral methods ([6]). The axisymmetric steady base flow undergoes a first bifurcation at the critical Reynolds number $\mathrm{Re}_{c1} = 213$ for an $m = 1$ non-oscillating global mode $\hat{\mathbf{q}}_1^{\mathsf{A}}$ ($\sigma_i = 0$). The spatial structure of the associated eigenmode displays strong large-scale axial velocity disturbances under the form of a pair of counter-rotating streamwise vortices (not shown here). Figures 2(a) and (b) show the bifurcated flow at the supercritical Reynolds number $\mathrm{Re} = 250$, obtained by the superposition of an arbitrary amount of perturbation on the base flow. The vortices induce a loss of symmetry of the base flow and the wake is shifted in a given direction ($\theta = 0$ here, due to the chosen normalization of $\hat{\mathbf{q}}_1^{\mathsf{A}}$), whereas it remains symmetric with respect to $r = 0$ in the orthogonal plane ($\theta = \pi/2$).

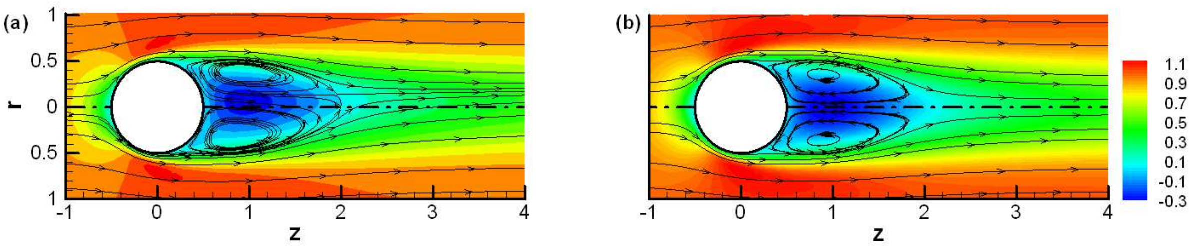

Figure 2: Contours of axial velocity w and streamlines for the total flow $\mathbf{q_0} + \epsilon^{1/2}\mathbf{q}_1^{\mathsf{A}}$ at $\mathrm{Re} = 250$ (arbitrary value of $\epsilon^{1/2}$). The dash-dotted line represents the symmetry axis of the base flow. (a) $\theta = 0, \pi$. (b) $\theta = \pi/2, 3\pi/2$.

A second bifurcation occurs at $\mathrm{Re}_{c2} = 281$ for an $m = 1$ oscillating global mode $\hat{\mathbf{q}}_1^{\mathsf{B}}$ of frequency $\sigma_i = 0.699$. The corresponding Strouhal number $St = fD/U_\infty$ of 0.111 is in excellent agreement with the experimental frequency $St = 0.118$ measured at this transitional Reynolds number ([7]). The associated eigenmode exhibits the spatially periodic downstream structure characteristic of the oscillatory wake instability (figure 3), hence indicating that this mode leads the periodic vortex shedding phenomenon.

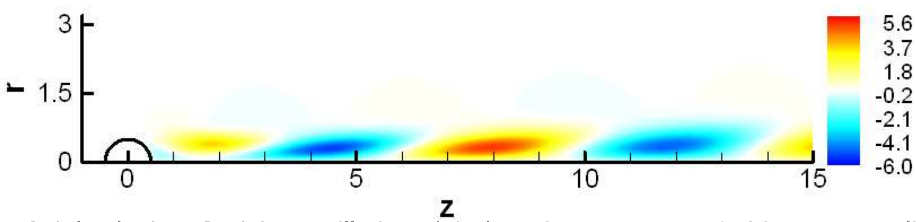

Figure 3: Axial velocity $\hat{w}_1^{\mathsf{B}}$ of the oscillating global mode at $\mathrm{Re} = 281$ (arbitrary normalization).

Adjoint analysis and non-normality The most amplified modes resulting from the global stability analysis, i.e. the leading global modes, govern the large-time dynamics of the flow. In this section, we use an adjoint analysis to investigate how this dynamics is affected by the small imperfections that are encountered in real flows. This point is of particular importance when considering experimental set-ups: for instance even the smallest sphere holding device induces perturbations that can be understood as local modifications of the base flow in the near wake.

Given the linear operator $\mathcal{M}$ defined in (4), the adjoint operator $\mathcal{M}^\dagger$ is defined as the operator such that, for any vectors $\hat{\mathbf{q}}$ and $\hat{\mathbf{q}}^\dagger$ fulfilling respective appropriate boundary conditions,

$$\langle \hat{\mathbf{q}}^\dagger, \mathcal{M}.\hat{\mathbf{q}} \rangle = \langle \mathcal{M}^\dagger.\hat{\mathbf{q}}^\dagger, \hat{\mathbf{q}} \rangle \,. \tag{5}$$

where $\langle \, , \rangle$ denotes the usual complex scalar product on Ω, *i.e.* $\langle \hat{\mathbf{q}}^1, \hat{\mathbf{q}}^2 \rangle = \int_\Omega \hat{\mathbf{q}}^{1T} \hat{\mathbf{q}}^2 d\Omega$. It can be shown that $\mathbf{q}^\dagger$ is solution of the eigenvalue problem

$$\mathcal{M}^\dagger.\hat{\mathbf{q}}^\dagger = \overline{\sigma} \mathcal{N}.\hat{\mathbf{q}}^\dagger \tag{6}$$

where $\overline{\sigma}$ is the complex conjugate of σ.

Figure 4 shows the adjoint axial velocity and adjoint pressure distributions for the oscillating adjoint global mode $\hat{\mathbf{q}}_1^{\mathsf{B}\dagger}$. We find very similar distributions for the non-oscillating adjoint global mode $\hat{\mathbf{q}}_1^{\mathsf{A}\dagger}$ (not shown here). Due to the non-normality of the operator $\mathcal{M}$, the adjoint global mode is located slightly upstream of the sphere and mainly within the recirculating area (marked by the thick solid line), whereas the associated global mode is located downstream of the sphere and extends down to large streamwise positions. The adjoint mode can be interpreted in terms of receptivity of the base flow to a volumic forcing, given by the velocity component $\hat{\mathbf{u}}^\dagger$, and to a boundary forcing, given at leading order by the wall pressure component $\hat{p}^\dagger$ ([2]). The adjoint analysis is therefore of particular interest in the elaboration of efficient control strategies (base-bleed, for instance) as we find that the adjoint axial velocity is concentrated within and at the periphery of the recirculating area, whereas the adjoint pressure peaks at the separation point. It can also be shown that it is possible to estimate the receptivity of the base flow to local modifications by considering the cooperation between a global mode and its adjoint global mode ([5], not shown here).

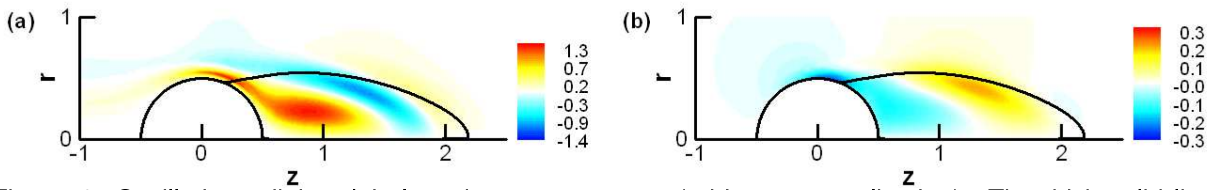

Figure 4: Oscillating adjoint global mode at $\mathrm{Re}_{c2} = 281$ (arbitrary normalization). The thick solid line marks the limit of the recirculating area. (a) Adjoint axial velocity $\hat{w}_1^{\mathsf{B}\dagger}$. (b) Adjoint pressure $\hat{p}_1^{\mathsf{B}\dagger}$.

Global weakly non-linear analysis In this section, we model the base flow undergoing two successive bifurcations by an extended dynamical system undergoing a multiple codimension bifurcation at the critical Reynolds number $\mathrm{Re}_{c2} = 281$. This assumption holds at leading order because both critical Reynolds numbers are close one from the other, so that $\xi = \mathrm{Re}_{c1}^{-1} - \mathrm{Re}_{c2}^{-1}$ is a small parameter of the problem. Substitution of the asymptotic expansion

$$\mathbf{q} = \mathbf{q}_0 + \epsilon^{1/2}\mathbf{q}_1 + \epsilon\mathbf{q}_2 + \epsilon^{3/2}\mathbf{q}_3 + ... \tag{7}$$

in the governing equations, where ϵ is the small parameter $\epsilon = \mathrm{Re}_{c2}^{-1} - \mathrm{Re}^{-1}$, leads to a series of equations of successive order $\epsilon^{i/2}$. At order 0, we find the non-linear equation specifying that $\mathbf{q}_0$ is a steady solution of the Navier-Stokes equations at the critical Reynolds number Re_{c2}. At order 1, we obtain the homogeneous linear equation specifying that $\mathbf{q}_1$ may be taken as a superposition of global modes of the steady flow field $\mathbf{q}_0$ at Re_{c2}. We can therefore choose $\mathbf{q}_1$ as the superposition of the marginal eigenmodes existing at the critical Reynolds number, each mode being multiplied by some complex scalar amplitude. Note that three global modes are to be considered, *i.e.* the system undergoes a codimension-three bifurcation: one mode for the first steady bifurcation and two superimposed modes of frequencies $\pm\sigma_i$ for the unsteady bifurcation. $\hat{\mathbf{q}}_1$ can therefore be written in the form

$$\hat{\mathbf{q}}_1 = \left(A\hat{\mathbf{q}}_1^{\mathsf{A}} + B^{+}\hat{\mathbf{q}}_1^{\mathsf{B}+} e^{\sigma t} + B^{-}\hat{\mathbf{q}}_1^{\mathsf{B}-} e^{\overline{\sigma} t} \right) e^{i\theta} + \text{ c.c.} \tag{8}$$

where A is the complex amplitude of the non-oscillating mode $\hat{\mathbf{q}}_1^{\mathsf{A}}$, and B^{+} (resp. B^{-}) is that of the oscillating mode $\hat{\mathbf{q}}_1^{\mathsf{B}+}$ (resp. $\hat{\mathbf{q}}_1^{\mathsf{B}-}$) of frequency σ_i (resp. $-\sigma_i$). At orders 2 and 3, we obtain inhomogeneous linear equations that can be understood as the harmonic linearized Navier-Stokes operator about $\hat{\mathbf{q}}_0$ forced by terms involving quantities of lower orders, which have therefore been determined. The homogeneous operator is non-degenerate at order 2 but degenerate at order 3, where the Fredholm alternative is used and compatibility conditions are applied, yielding a system of Stuart-Landau amplitude equations for the complex amplitudes (A, B^+, B^-). Although this system can be calculated from the full equations, it arises naturally by considering that invariance under the transformation $(A, B^+, B^-) \longrightarrow (A, B^+, B^-)e^{i\varphi}$ is required, where φ is an arbitrary phase. The system of coupled amplitude equations finally reads

$$dA/dt = \epsilon\lambda_{\mathsf{A}} A - \epsilon A \left(\mu_{\mathsf{A}} |A|^2 + \nu_{\mathsf{A}} |B^+|^2 + \overline{\nu}_{\mathsf{A}} |B^-|^2\right) - \epsilon\chi_{\mathsf{A}} B^+ \overline{B^- A} \quad (9a)$$

$$dB^+/dt = \epsilon\lambda_{\mathsf{B}} B^+ - \epsilon B^+ \left(\mu_{\mathsf{B}} |B^+|^2 + \nu_{\mathsf{B}} |B^-|^2 + \eta_{\mathsf{B}} |A|^2\right) - \epsilon\chi_{\mathsf{B}} B^- A^2 \quad (9b)$$

$$dB^-/dt = \epsilon\lambda_{\mathsf{B}} B^- - \epsilon B^- \left(\mu_{\mathsf{B}} |B^-|^2 + \nu_{\mathsf{B}} |B^+|^2 + \eta_{\mathsf{B}} |A|^2\right) - \epsilon\chi_{\mathsf{B}} B^+ \overline{A}^2 . \quad (9c)$$

By use of the compatibility conditions, the coefficients of system (9) arise as scalar products between the adjoint global modes computed in the previous section and the forcing terms of order 3 of appropriate complex amplitude. For instance, $\mu_{\mathsf{A}} = \langle \hat{\mathbf{q}}^{\dagger}_{1\mathsf{A}}, \hat{\mathbf{f}}_3^{A|A|^2} \rangle$ where $\hat{\mathbf{f}}_3^{A|A|^2}$ is the forcing term of complex amplitude $A|A|^2$, arising from the non-linear interaction of $\hat{\mathbf{q}}_1^{\mathsf{A}}$ with the order 2 mode of amplitude $|A|^2$ and of $\overline{\hat{\mathbf{q}}_1^{\mathsf{A}}}$ with the order 2 mode of amplitude A^2, *i.e.*

$$\hat{\mathbf{f}}_3^{\,A|A|^2} = -C[\hat{\mathbf{q}}_1^{\mathsf{A}}, \hat{\mathbf{q}}_2^{|\mathsf{A}|^2}] - C[\overline{\hat{\mathbf{q}}_1^{\mathsf{A}}}, \hat{\mathbf{q}}_2^{\mathsf{A}^2}] . \quad (10)$$

Numerically, we obtain

$\lambda_{\mathsf{A}} = 147$	$\lambda_{\mathsf{B}} = 200 - 8.45$ i
$\mu_{\mathsf{A}} = 16.2$	$\mu_{\mathsf{B}} = 0.355 + 0.0301$ i
$\nu_{\mathsf{A}} = 0.415 - 0.0155$ i	$\nu_{\mathsf{B}} = 0.308 + 0.168$ i
	$\eta_{\mathsf{B}} = 20.1 - 1.83$ i
$\chi_{\mathsf{A}} = 0.0165$	$\chi_{\mathsf{B}} = 8.55 + 1.90$ i .

We will discuss the formation of slowly rotating horseshoe vortices as a particular solution of this complex system, that may even admit chaotic solutions (3 degrees of freedom for amplitudes and 3 others for phases).

Conclusion The first and second bifurcations of the steady axisymmetric wake of the sphere is investigated numerically in the framework of the global linear stability. The adjoint problem is solved as a step towards a weakly non-linear analysis and the adjoint global modes are discussed in terms of receptivity to flow control and base flow modifications. A system of coupled amplitude equations is derived for a dynamical system undergoing a codimension-three bifurcation, whose resolution is expected to provide useful information for the description of the early stage of the periodic regime.

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