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The dipolar field of rotating bodies in two dimensions

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The fluid velocity far from a translating body in two-dimensional irrotational flow is generally dipolar. This is a classical result. Here we ask when the dipolar component vanishes. Lamb (1945, § 126) provides symmetry conditions on the virtual mass tensor for this to be the case. We show that these conditions are not necessary and obtain the sufficient and necessary condition in terms of the shape of the body using conformal maps. Some explicit examples are constructed based on this condition.

1. Introduction

The classical theory of the motion of a solid through an inviscid fluid was developed by Kirchhoff and subsequently by Thomson and Tait, and is summarized in Lamb (1945, Chapter VI). The coupled fluid–body system, which in its primary formulation requires solving the Euler equations and Newton's equations for the body, can be replaced by a system of ordinary differential equations that take into account the forces and couples exerted on the solid by the fluid. The flow is taken to be irrotational, which is the case if the body starts from rest, for example. This approach has been extensively pursued in ship hydrodynamics in particular (e.g. Newman 1977).

The problem reduces to the computation of the kinetic energy of the fluid, which may be written as

$$T = \tfrac{1}{2} m_{ij} V_i V_j, \tag{1.1}$$

where m_{ij} is a 6×6 tensor known as the added-mass tensor and $\boldsymbol{V}^T = (\boldsymbol{U}^T \boldsymbol{\Omega}^T)$ is an extended 6×1 vector combining the velocity $\boldsymbol{U}$ and angular velocity $\boldsymbol{\Omega}$ of the body referred instantaneously to a frame moving with the body. One can show that m_{ij} can be written as

$$m_{ij} = \rho \int_C \varphi_i \frac{\partial \varphi_j}{\partial n}\, \mathrm{d}l, \tag{1.2}$$

where the suffices correspond to the decomposition of the velocity potential according to $\varphi = \sum_{j=1}^{6} V_j \varphi_j$. The normal vector $\boldsymbol{n}$ points out of the fluid into the body. Integrating by parts shows that m_{ij} is a symmetric matrix. On physical grounds, it must also be a positive semi-definite tensor.

We are interested in the far-field behaviour of the flow induced by the motion of a body. We restrict ourselves to two dimensions and assume that the circulation around the body is zero. In this case only the elements m_{11}, m_{12}, m_{22}, m_{16}, m_{26} and m_{66} (and the corresponding elements obtained by symmetry) remain.

The far-field behaviour of the flow potential is of particular interest to study the forces applied on the solid body as these are directly related to the dipole moment (see for example Saffman 1992).

2. The far-field potential and added-mass coefficients

The velocity potential is found by solving a classical Dirichlet problem:

$$\nabla^2\varphi = 0 \quad \text{with boundary condition} \quad \begin{cases} \nabla\varphi \cdot \boldsymbol{n} = (\boldsymbol{U} + \Omega \boldsymbol{e}_3 \times \boldsymbol{x}) \cdot \boldsymbol{n} \text{ on the body,} \\ \nabla\varphi \to 0 \text{ at infinity;} \end{cases} \tag{2.1}$$

$\boldsymbol{U}$ is now a two-dimensional vector.

Using the free-space Green's function to solve for φ in (2.1), the far-field behaviour of φ may be obtained:

$$\varphi(\boldsymbol{x}_0) = -\frac{1}{2\pi}\log r_0 \int_C \frac{\partial\varphi}{\partial n}\,\mathrm{d}l - \frac{\boldsymbol{x}_0}{2\pi r_0^2}\cdot\int_C \left(\varphi\boldsymbol{n} - \frac{\partial\varphi}{\partial n}\boldsymbol{x}\right)\mathrm{d}l + O\left(r_0^{-2}\right). \tag{2.2}$$

Since we are considering only rigid bodies the first term, which is proportional to the rate-of-change of the body's area, is zero (this can be seen by rewriting it in terms of a single-valued streamfunction). Hence we have the result for large r_0:

$$\varphi(\boldsymbol{x}_0) \sim -\frac{\boldsymbol{x}_0}{2\pi r_0^2}\cdot\int_C \left(\varphi\boldsymbol{n} - \frac{\partial\varphi}{\partial n}\boldsymbol{x}\right)\mathrm{d}l. \tag{2.3}$$

The integral $\boldsymbol{I}$ is a two-component vector. The first term is the virtual momentum of the body $\boldsymbol{I}_b$ (Saffman 1992, §4.1, who sets the density to unity); we call the second term $\boldsymbol{I}_c$. The leading-order behaviour is dipolar unless $\boldsymbol{I}$ vanishes.

For the case of pure translation, we write $\varphi = U_j\varphi_j$. Then the virtual momentum can be written as

$$U_j\int_C \varphi_j n_i\,\mathrm{d}l = U_j\int_C \varphi_j\frac{\partial\varphi_i}{\partial n}\,\mathrm{d}l = m_{ij}U_j, \tag{2.4}$$

where we have used the boundary condition $\partial\varphi_j/\partial n = n_j$ on the boundary and set the density ρ to unity. We have recovered the elements of the virtual-mass tensor m_{ij} for $1 \leqslant i, j \leqslant 2$. Again, using the boundary condition, I_{ci} becomes

$$-U_j\int\frac{\partial\varphi_j}{\partial n}x_i\,\mathrm{d}l = -U_j\int_C x_i n_j\,\mathrm{d}l = U_j\int_S \delta_{ij}\,\mathrm{d}S = AU_i \tag{2.5}$$

on using the divergence theorem (there is a change of sign because the normal is directed into the body). Here A is the area of the body. Hence

$$I_i = (m_{ij} + A\delta_{ij})U_j. \tag{2.6}$$

This is a standard result: see Lamb (1945; §121a) for the analogous three-dimensional equivalent, and Newman (1977, §4.14 equations 128 and 130) for the two- and three-dimensional results.

We now consider pure rotation of the body. We have the boundary condition $\partial\varphi_6/\partial n = \epsilon_{jk}n_jx_k$, where ϵ_{ij} is the alternating tensor. Now $I_{bi} = m_{i6}$, while I_{ci} becomes

$$-\Omega\epsilon_{jk}\int_C n_jx_kx_i\,\mathrm{d}l = \Omega\epsilon_{jk}\int_A(\delta_{jk}x_i + x_k\delta_{ij})\mathrm{d}S = \Omega\epsilon_{ik}A\bar{x}_k, \tag{2.7}$$

where $\bar{\boldsymbol{x}}$ is the geometric centre of the body. In this case

$$I_i = (m_{i6} + \epsilon_{ij} A \bar{x}_j)\Omega. \tag{2.8}$$

If we fix the position of the centre of the axes at the geometric centre of the body so that $\bar{\boldsymbol{x}} = \boldsymbol{0}$, we obtain the following condition: if the virtual-mass tensor coefficients m_{16} and m_{26} vanish, the velocity potential due to a body rotating about its geometric centre in two dimensions has no dipolar component. We now ask: when do m_{16} and m_{26} vanish?

The symmetry properties of the virtual mass tensor are discussed in Lamb (1945; § 126), where it is pointed out that "as might be anticipated from the complexity of the question, the physical meaning of the results is not easily grasped". Considering the two-dimensional body as being extended indefinitely in the third dimension, m_{16} and m_{26} vanish in the following cases (the headings are taken from Lamb):

3°. If the body has two axes of symmetry at right angles.

4° and 5°. If the body is a circle or a regular polygon.

7°. If the body is symmetric under a rotation by a right angle.

It is important to point out that these are sufficient, but not necessary, conditions. Can we find a sufficient and necessary condition? Since this is a problem concerning shapes of two-dimensional bodies, it is natural to use conformal mapping theory.

3. Complex-variable formulation

Suppose a two-dimensional object is translating with speed U at an angle α to the x-axis and is rotating about the origin with angular velocity Ω. The irrotational flow exterior to the body can be described by a complex potential

$$w(z) = \phi + \mathrm{i}\psi \tag{3.1}$$

where $z = x + \mathrm{i}y$, ϕ is the velocity potential and ψ is the streamfunction. ψ is also harmonic in the fluid region and, like ϕ, it satisfies a classical Dirichlet boundary value problem: in the fluid,

$$\nabla^2\psi = 0, \tag{3.2}$$

while on the body surface,

$$\psi = -\frac{\Omega}{2} z\bar{z} - \frac{\mathrm{i}U\mathrm{e}^{-\mathrm{i}\alpha}}{2} z + \frac{\mathrm{i}U\mathrm{e}^{\mathrm{i}\alpha}}{2}\bar{z}, \tag{3.3}$$

and $\psi = o(z)$ at infinity.

The Dirichlet boundary value problem is known to be conformally invariant. Therefore we invoke the Riemann mapping theorem which states that there exists a conformal mapping, $z = f(\zeta)$ say, from the interior of the unit disk in a complex ζ-plane to the unbounded region exterior to any simply connected body. Without loss of generality, we can also suppose that $\zeta = 0$ maps to $z = \infty$ and that, near $\zeta = 0$, $f(\zeta)$ has a convergent Laurent expansion of the form

$$z = f(\zeta) = \frac{a_{-1}}{\zeta} + \sum_{n=0}^{\infty} a_n \zeta^n \tag{3.4}$$

where, by a further degree of freedom in the mapping theorem, we are free to take a_{-1} to be real. We can also introduce the analytic function of ζ given by

$$W(\zeta) = w(z(\zeta)) \tag{3.5}$$

which is analytic everywhere in $|\zeta| \leqslant 1$.

Now the boundary value problem for ψ takes the form

$$\mathrm{Re}[-\mathrm{i}W(\zeta)] = \psi = -\frac{\Omega}{2}z\bar{z} - \frac{\mathrm{i}U\mathrm{e}^{-\mathrm{i}\alpha}}{2}z + \frac{\mathrm{i}U\mathrm{e}^{\mathrm{i}\alpha}}{2}\bar{z} \qquad \text{on } |\zeta| = 1. \tag{3.6}$$

Since $W(\zeta)$ must be analytic in $|\zeta| \leqslant 1$, this is just the Schwarz problem for the unit disk and its solution is given by the Poisson integral formula:

$$-\mathrm{i}W(\zeta) = \frac{1}{2\pi\mathrm{i}}\oint_{|\zeta'|=1} \frac{\mathrm{d}\zeta'}{\zeta'}\left(\frac{\zeta'+\zeta}{\zeta'-\zeta}\right)\left[-\frac{\Omega}{2}z(\zeta')\overline{z(\zeta')} - \frac{\mathrm{i}U\mathrm{e}^{-\mathrm{i}\alpha}}{2}z(\zeta') + \frac{\mathrm{i}U\mathrm{e}^{\mathrm{i}\alpha}}{2}\overline{z(\zeta')}\right]. \tag{3.7}$$

For $|\zeta| \leqslant 1$, $W(\zeta)$ has a convergent Taylor expansion of the form

$$W(\zeta) = W_0 + W_1\zeta + W_2\zeta^2 + \cdots \tag{3.8}$$

and formulae for the coefficients $\{W_n | n = 0, 1, \ldots\}$ can be readily generated from (3.7). As $z \to \infty$, $w(z)$ has a far-field expansion of the form

$$w(z) = w_0 + \frac{w_1}{z} + \frac{w_2}{z^2} + \cdots \tag{3.9}$$

where w_1 is the far-field dipole moment. From (3.4) and (3.5) it is clear that requiring w_1 to vanish is equivalent to setting $W_1 = 0$. From (3.7) the latter condition is found to be equivalent to

$$\oint_{|\zeta'|=1} \frac{\mathrm{d}\zeta'}{\zeta'^2}\left[-\frac{\Omega}{2}z\bar{z} - \frac{\mathrm{i}U\mathrm{e}^{-\mathrm{i}\alpha}}{2}z + \frac{\mathrm{i}U\mathrm{e}^{\mathrm{i}\alpha}}{2}\bar{z}\right] = 0. \tag{3.10}$$

We now restrict attention to a class of body shapes having analytic boundary shapes with no corners. We avoid corners since the velocity field is necessarily infinite at such points when there is no circulation around the body to regularize such flow singularities. In such a case, the Laurent expansion (3.4) is convergent for $|\zeta| \leqslant 1$. We find the area of the body A from

$$A = -\frac{1}{2\mathrm{i}}\oint_{|\zeta|=1} \bar{f}(\zeta^{-1})f'(\zeta)\,\mathrm{d}\zeta \tag{3.11}$$

or, making use of (3.4),

$$a_{-1}^2 - \sum_{n=1}^{\infty} n|a_n|^2 = \pi^{-1}A. \tag{3.12}$$

The case of a finite flat plate, which has zero area, is excluded by our constraint that the body boundary has no corners. More generally, we require the area to be positive so that $a_{-1}^2 - |a_1|^2 > 0$.

3.1. *Pure translation*

If $\Omega = 0$ (with $U \neq 0$) condition (3.10) reduces to

$$a_1\mathrm{e}^{-\mathrm{i}\alpha} - a_{-1}\mathrm{e}^{\mathrm{i}\alpha} = 0. \tag{3.13}$$

Since this implies that $a_{-1} = |a_1|$, which is inconsistent with the fact that $a_{-1}^2 - |a_1|^2 > 0$, we conclude immediately that the dipole moment of a simply connected body (with non-zero area and analytic boundary) in pure translation can never vanish.

3.2. *Rotating bodies*

Suppose first that $\Omega \neq 0$. If the body is in pure rotation, so that $U=0$, (3.10) reduces to

$$\oint_{|\zeta|=1} \frac{\mathrm{d}\zeta}{\zeta^2} f(\zeta)\overline{f(\zeta)} = 0. \tag{3.14}$$

If $U \neq 0$, we may write

$$g(\zeta) \equiv f(\zeta) + \frac{\mathrm{i}U\mathrm{e}^{\mathrm{i}\alpha}}{\Omega}, \tag{3.15}$$

and obtain (3.14) with $g(\zeta)$ replacing $f(\zeta)$. The mapping $g(\zeta)$ is also a conformal mapping from the unit disk to the exterior of the same body but with its centroid simply shifted. We conclude that it is enough to consider the case where a body is in pure solid-body rotation about the origin (i.e. $U=0$) but with different centroid positions.

Introduce the function

$$h(\zeta) = \frac{a_{-1}}{\zeta} + \sum_{n=1}^{\infty} a_n \zeta^n \tag{3.16}$$

so that $f(\zeta) = a_0 + h(\zeta)$. Condition (3.14) can then be written in the form

$$\left[\oint_{|\zeta|=1} \frac{\mathrm{d}\zeta}{\zeta^2} \overline{h}(\zeta^{-1})\right] a_0 + \left[\oint_{|\zeta|=1} \frac{\mathrm{d}\zeta}{\zeta^2} h(\zeta)\right] \overline{a_0} = -\oint_{|\zeta|=1} \frac{\mathrm{d}\zeta}{\zeta^2} h(\zeta)\overline{h}(\zeta^{-1}), \tag{3.17}$$

or

$$a_{-1}a_0 + a_1\overline{a_0} = r_0 \tag{3.18}$$

where

$$r_0 \equiv -\frac{1}{2\pi \mathrm{i}} \oint_{|\zeta|=1} \frac{\mathrm{d}\zeta}{\zeta^2} h(\zeta)\overline{h}(\zeta^{-1}). \tag{3.19}$$

Equation (3.18), and its complex conjugate, can always be solved uniquely for a_0 owing to the fact that, from (3.12), we know that $a_{-1}^2 - |a_1|^2 > 0$. If a_0 is chosen in this way, the deduction is that, for a body of any (analytic) shape and for any Ω, there exists a special location of its centroid such that the far-field dipole moment vanishes.

As just discussed, it is enough to consider the class of purely rotating bodies for which some choice of the centroid position relative to the origin is made. A natural first choice is to fix the centroid location to coincide with the origin so we now restrict to this case and examine whether we can find bodies with vanishing far-field dipole moment. It is straightforward to check that condition (3.14) is equivalent to the condition

$$a_{-1}\overline{a_0} + a_0\overline{a_1} + a_1\overline{a_2} + \cdots = \sum_{n=-1}^{\infty} a_n\overline{a_{n+1}} = 0; \tag{3.20}$$

let us call this condition (D) for dipole. First, let us retrieve the known results of Lamb from this new perspective. The conformal map to a circle is simply

$$z(\zeta) = \frac{a_{-1}}{\zeta}. \tag{3.21}$$

This clearly satisfies (D). Now consider a body with centroid at the origin which has two axes of symmetry at right angles. Without loss of generality, take these axes to be the real and imaginary axes. It is easy to establish that the conformal map to such a body has the general form

$$z(\zeta) = \frac{a_1}{\zeta} + a_1\zeta + a_3\zeta^3 + a_5\zeta^5 + \cdots \tag{3.22}$$

where all the coefficients are real. Such a map also satisfies (D). A body with an m-polygonal symmetry (with $m \geqslant 2$) centred at the origin can be shown to correspond to a conformal mapping of the form

$$z(\zeta) = \frac{a_{-1}}{\zeta} + a_{m-1}\zeta^{m-1} + a_{2m-1}\zeta^{2m-1} + a_{3m-1}\zeta^{3m-1} + \cdots \tag{3.23}$$

which, again, satisfies condition (D).

4. Existence of solutions

As mentioned in the previous section, it is always possible to choose the centre of rotation (by solving (3.18) for a_0) so that (D) is satisfied and this solution is unique. Here we look for particular cases where this centre of rotation and the centroid coincide and our problem can be formulated as:

Find solutions other than Lamb's solutions of the form (3.4) with f univalent that satisfy (D) and whose centroid is at the origin.

We distinguish two new classes of solutions: solutions that satisfy (D) trivially (i.e. there are no consecutive terms in the Laurent series and all the terms in (D) are zero), and solutions that satisfy (D) non-trivially. From the definition of the centroid z_c,

$$Az_c = \frac{1}{2\mathrm{i}} \oint_{|\zeta|=1} f(\zeta)\overline{f}(\zeta^{-1}) f'(\zeta)\,\mathrm{d}\zeta, \tag{4.1}$$

the centroid condition (C) can be expressed for the class of mappings (3.4) as

$$\sum_{p=-1}^{\infty} \sum_{r=-1}^{\infty} r a_p \overline{a_{p+r}} a_r = 0, \tag{4.2}$$

where $-1 \leqslant p + r$. Note that the sum in r starts at $r = 2$ if (D) is satisfied. We truncate the sums at N, so that $-1 \leqslant p + r \leqslant N$.

One can construct solutions for example by expressing a_{-1} in terms of other a_i from (D) and then substituting into (C). This leads to an equation for a_0 and $\overline{a_0}$. Taking the conjugate gives a quadratic equation for a_0. There are other possibilities, all of which lead to a polynomial equation for one of the a_i in terms of $N - 2$ of them. Unfortunately, applying this approach naively leads to self-intersecting shapes and non-univalent maps, which are unphysical. We need to verify *a posteriori* that the mapping is univalent (i.e. one-to-one) so that it corresponds to a conformal map to a physically admissible body shape.

4.1. *Simple curves*

There is a well-known condition for a plane curve to be non-self-intersecting, or simple (Kühnel 2006):

THEOREM 1. *(Hopf) For a simply closed regular curve, we have*

$$U_c = \frac{1}{2\pi} \oint \kappa \,\mathrm{d}s = \pm 1, \tag{4.3}$$

where κ is the curvature.

Then, with $\zeta = \mathrm{e}^{\mathrm{i}t}$ and $z = f(\zeta)$ as before, some algebra gives

$$U_c = 1 + \oint \frac{f''(\zeta)}{f'(\zeta)}\,\mathrm{d}\zeta. \tag{4.4}$$

By the argument principle, $U_c = 1 + Z - P = -1 + Z$ here, where Z and $P = 2$ are the numbers of zeros and poles (counted with multiplicity) inside the unit circle $|\zeta| = 1$. (The case $Z = 2$ is not relevant.) Hence, the shape is simple and regular if there are no zeros of $f'(\zeta)$ inside $|\zeta| = 1$.

Another way to understand this condition is by means of the implicit function theorem: given the mapping $z = f(\zeta)$ the condition $f'(\zeta) \neq 0$ is a necessary condition for it to be (locally) uniquely invertible for $\zeta = g(z)$ where g denotes the inverse function f^{-1}.

4.2. *Restricted class of shapes*

In an attempt to construct explicit examples of body shapes with vanishing dipole moment, beyond the highly symmetric ones identified by Lamb, we focus on a restricted class of possible shapes. We consider the class of shapes given by conformal mappings whose Laurent series expansions have only a finite number of terms so that, for some integer N,

$$f'(\zeta) = -\frac{1}{\zeta^2} + a_1 + 2a_2\zeta + \cdots + (N+1)a_{N+1}\zeta^N = \frac{1}{\zeta^2}(-1 + b_2\zeta^2 + b_3\zeta^3 + \cdots + b_{N+2}\zeta^{N+2}). \tag{4.5}$$

Then, we are led to ask: when do the zeros of a polynomial $p = -1 + b_1\zeta + \cdots + b_{N+2}\zeta^{N+2}$ lie inside the unit circle? To answer this question, consider the following theorem (Henrici 1974; Marden 1985):

THEOREM 2. *(Schur–Cohn algorithm) All the zeros of a polynomial p of degree n lie outside the closed unit disk if and only if $\gamma_k > 0$, $k = 1, 2, \ldots, N$.*

In the full statement of this theorem the quantities $\{\gamma_k | k = 1, \ldots, N\}$ are given by explicit expressions in terms of the coefficients of the polynomial. Unfortunately these expressions rapidly become very complicated. So, while these conditions can easily be checked *a posteriori*, it is generally no simpler than actually computing the location of the zeros.

The non-vanishing of $f'(\zeta)$ is a necessary condition for the mapping to be univalent (and, hence, physically admissible). However, this condition on the zeros of the derivative is not sufficient for univalence and a mapping which satisfies it may still fail to be univalent.

4.3. *Univalence*

A complex function $f(z)$ is univalent in a domain Ω if $f(z_1) = f(z_2)$ implies $z_1 = z_2$. The theory of univalent functions is a vast topic (for a good overview, see Goodman 1983) but, since we only require to find specific examples of shapes satisfying our particular problem, we are only interested here in very specific cases. The following result is relevant to our restricted class of maps (Goodman 1983, Exercise 9.74):

THEOREM 3. *(Aksent'ev) For mappings of the form considered, if $|p + 1| \leqslant 1$ in the unit disk, the mapping is univalent or corresponds to a mapping onto an ellipse.*

This is a sufficient condition (the elliptical case can be excluded immediately for the cases we consider). Actually, the following necessary conditions exist but we shall not make use of them:

THEOREM 4. *(Loewner) If the mapping f above is univalent, then $|p| \geqslant 1 - |\zeta|^2$ and $|p| \leqslant (1 - |\zeta|^2)^{-1}$ in the unit disk.*

4.4. *Proof of existence of solutions*

We can show that non-trivial solutions exist using the property of polynomials that their zeros are continuous functions of the coefficients. Hence if we start with a symmetric solution and perturb the coefficients by a small amount, we know that the zeros of p can only move by a small amount. If we perturb the coefficients in such a way that (C) and (D) are still satisfied, we have proved existence of solutions to our problem.

The argument can be formalized as follows (this is not the most general statement). Consider a perturbation of a symmetric Lamb solution with $m > 1$ such that

$$f(\zeta) = \frac{1+\alpha_{-1}}{\zeta} + \alpha_0 + \alpha_1\zeta + \cdots + \alpha_{m-1}\zeta^{m-1} + (a_m + \alpha_m)\zeta^m \tag{4.6}$$

and the α_i are small. The root polynomial is

$$p = -(1+\alpha_{-1}) + \alpha_1\zeta^2 + 2\alpha_2\zeta^3 + \cdots + (m-1)\alpha_{m-1}\zeta^m + m(a_m+\alpha_m)\zeta^{m+1}. \tag{4.7}$$

The symmetric solution with $\alpha_i = 0$ satisfies (D) and (C) automatically. The roots of p are given by $\rho^{m+1} = 1/ma_m$ so we require $1 > m|a_m|$ (adding the area condition leads to an extra condition).

Linearizing (D) and (C) gives a system of equations with the trivial solution: $\alpha_0 = \alpha_{m-1} = 0$. We can now expand (4.7) with $\zeta = \rho + \eta$ and solve the resulting linear equation for η. We find $\eta = O(\alpha)$, so the roots stay outside the unit circle. We can even make η zero by choosing the α_i appropriately. Similarly the univalence of the mapping changes continuously and is hence preserved in this argument.

5. Explicit solutions

We now present some concrete examples of generalized body shapes (we use the word 'generalized' to refer to body shapes with vanishing dipole moment which lie outside the classes of shapes already identified by Lamb). Of course, we can safely conclude that there exist infinitely many generalized shapes and it is not our aim to classify them. Instead we present examples of trivial (when all terms in (D) are zero) and non-trivial solutions. One can show that for $N \leqslant 2$, the only solutions are ellipses. Hence we concentrate on $N > 2$. We do not enforce the area condition, but instead take $a_{-1} = 1$ and require the area to be positive. Then the area condition can be enforced *a posteriori* by rescaling.

5.1. *Generalized trivial solutions*

There are no such solutions for $N = 3$. For $N = 4$, there are two possibilities: either $a_0 = a_2 = a_3 = 0$ or $a_0 = a_1 = a_3 = 0$, but the second case reduces to Lamb solutions. It can be shown that (C) is also satisfied trivially. The Schur–Cohn algorithm gives five relations between the two complex coefficients a_1 and a_4. If we take a_1 and a_4 to be real, corresponding to a subset of all possible shapes, we can find the sufficient domain in the (a_4, a_1)-plane explicitly. It is given by

$$\frac{1-16a_4^2}{32a_4^2}\left(1-\sqrt{1+64a_4^2}\right) < a_1 < 1 - 4|a_4|. \tag{5.1}$$

The necessary condition from Theorem 3 becomes

$$|a_1| + |a_4| \leqslant 1, \tag{5.2}$$

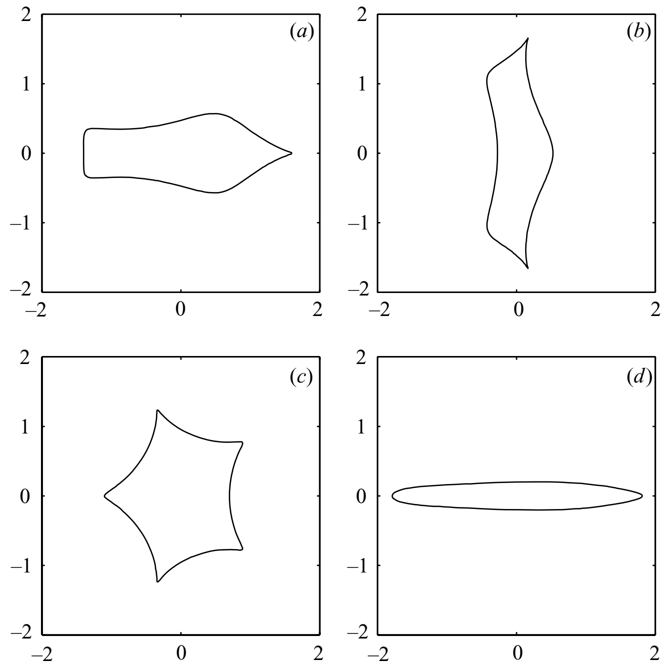

FIGURE 1. Generalized trivial solutions. The non-zero coefficients are $a_{-1} = 1$, $a_1 = (0.5, -0.6, -0.1, -0.8)$ and $a_4 = (0.1, 0.125, -0.2, 0.01)$ respectively.

which is simpler. These solutions all have positive area except at the points $a_1 = 0$, $a_4 = \pm 1$. The axes in the (a_1, a_4)-space are Lamb solutions. Some representative shapes are shown in figure 1. Although shapes (*a*) and (*b*) are close to the boundary of the region of feasibility and hence are close to having cusps, none of the shapes presented have strictly sharp corners. Shape (*b*) is outside the region given by (5.2) but is still univalent.

5.2. *Generalized non-trivial solutions*

We work with $N = 3$. One can show that all coefficients must be non-zero. Once again, we limit ourselves to real coefficients. There are a number of ways of solving (C) and (D); we choose to specify $a_{-1} = 1$ again and pick a_2 and a_3. One can then reduce the problem to a quadratic in a_1 and subsequently obtain a_0. The Schur–Cohn algorithm gives two simple results: $|a_3| < \frac{1}{3}$ and $|a_3| < \frac{1}{3}\sqrt{1 - 2|a_2|}$. The second is sharp for one of the roots of the quadratic when $a_3 < 0$, but the other $\{\gamma_k\}$ are very difficult to use. The univalence condition is hard to find in closed form. We limit ourselves to showing examples of feasible shapes in figure 2. It appears that only one of the solutions of the quadratic equation for a_1 leads to a univalent mapping.

6. Conclusion

We have shown that, beyond the highly symmetrical shapes already discussed by Lamb, there exist generalized body shapes with non-zero area, analytic boundaries and with centroid at the origin which also have vanishing dipole moment. We call them trivial when the dipole moment condition is identically zero and otherwise non-trivial. The existence of such generalized shapes is of theoretical interest. Moreover, we have established their existence constructively by employing the powerful machinery

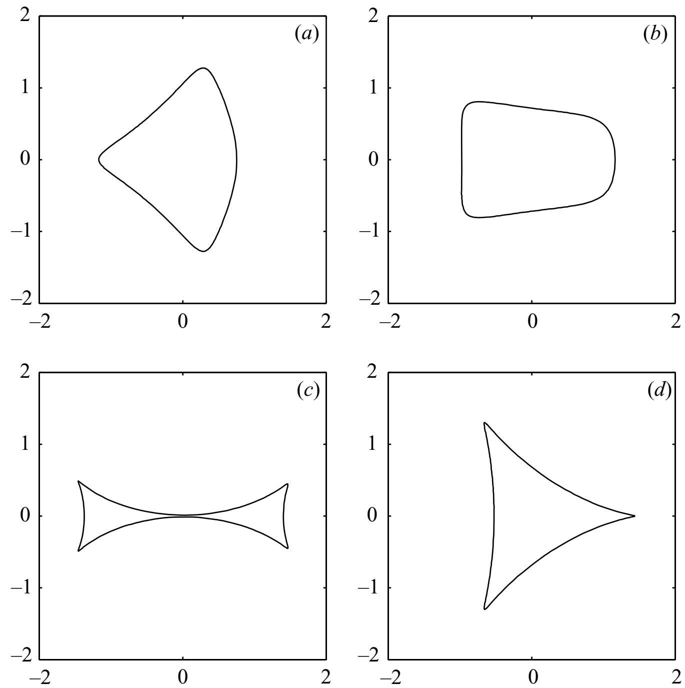

FIGURE 2. Generalized non-trivial solutions. The coefficients are $a_{-1} = 1$, $a_0 = (-0.0090, -0.0058, -0.0046, 0.0082)$, $a_1 = (-0.1386, 0.1683, 0.6873, -0.0473)$, $a_2 = (-0.2, 0.1, 0.02, 0.45)$, $a_3 = (0.1, -0.2, -0.3, 0.03)$. For these shapes a_{-1}, a_2 and a_3 are exact; the others are obtained numerically.

of conformal mapping theory and solving the resulting transcendental relation (D) (at least for a restricted class of possible shapes). It is clear from our approach that the existence of these generalized shapes would be difficult, almost impossible, to infer from general arguments based on symmetry.

We have proved the existence of generalized body shapes using the continuous dependence of the roots of polynomials on the polynomial's coefficients. We have then found solutions explicitly for simple forms of the mapping. The requirement that the mapping be univalent must be verified *a posteriori*. Necessary and sufficient conditions are derived and univalent solutions are found.

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