GEOPHYSICAL RESEARCH LETTERS, VOL. 36, L16804, doi:10.1029/2009GL039667, 2009

Click Here for Full Article

A model for the relationship between tropical precipitation and column water vapor

Caroline J. Muller,[1] Larissa E. Back,[1] Paul A. O'Gorman,[1] and Kerry A. Emanuel[1]

Received 18 June 2009; revised 22 July 2009; accepted 28 July 2009; published 25 August 2009.

[1] Several observational studies have shown a tight relationship between tropical precipitation and column-integrated water vapor. We show that the observed relationship in the tropics between column-integrated water vapor, precipitation, and its variance can be qualitatively reproduced by a simple and physically motivated two-layer model. It has previously been argued that features of this relationship could be explained by analogy with the theory of continuous phase transitions. Instead, our model explicitly assumes that the onset of precipitation is governed by a stability threshold involving boundary-layer water vapor. This allows us to explain the precipitation-humidity relationship over a broader range of water vapor values, and may explain the observed temperature dependence of the relationship. **Citation:** Muller, C. J., L. E. Back, P. A. O'Gorman, and K. A. Emanuel (2009), A model for the relationship between tropical precipitation and column water vapor, *Geophys. Res. Lett.*, *36*, L16804, doi:10.1029/2009GL039667.

1. Introduction

[2] Rainfall and column-integrated water vapor are closely related in the tropics [e.g., *Bretherton et al.*, 2004; *Peters and Neelin*, 2006; *Neelin et al.*, 2009; *Holloway and Neelin*, 2009]. The existence of a positive correlation between rainfall and humidity is unsurprising: high humidity can be both a cause and consequence of deep convection and rainfall. In fact, aspects of this relationship are an integral part of theories for explaining tropical phenomena including the MJO [e.g., *Bony and Emanuel*, 2005; *Khouider and Majda*, 2006; *Raymond*, 2000; *Raymond and Fuchs*, 2009], convectively-coupled waves [e.g., *Neelin and Yu*, 1994; *Kuang*, 2008] and hurricanes [e.g., *Emanuel*, 1995]. However, we do not at present have a full understanding of the mechanisms underlying the observed humidity-rainfall relationship. In this work, we propose a possible framework for understanding some key aspects of this relationship.

[3] Our work evolved from a discussion about the observations presented by *Neelin et al.* [2009] (hereinafter referred to as NPH). They show that instantaneous precipitation, P, on 25 $\times$ 25 km scales increases with column water vapor, w, with a sharp increase near a critical value of w and then a somewhat slower increase at higher w (Figure 1). The sharp increase (or pickup) is associated with a peak in precipitation variance near the critical w. NPH and a preceding study [*Peters and Neelin*, 2006] also argue that precipitation at high w follows a power law with a universal (temperature independent) power of about 0.21–0.26.

[4] NPH (and earlier work by these authors) use these results to argue that self-organized criticality (SOC) is a useful theoretical framework to explain the dependence of precipitation on water vapor. In their view, precipitation is a critical phenomenon, and the system self-organizes towards the critical point of the transition to strong precipitation. While this idea is interesting, the empirical evidence for SOC is essentially circumstantial. Until a clear physical mechanism is provided, the evidence for SOC from the humidity-rainfall relationship relies heavily on a power law fit to very high humidity values, at which observations are limited. It is also unclear how applicable a theory based on very high humidity and precipitation values is to understanding the relevant physics in more typical rainy conditions.

[5] In light of these issues, we wondered if a simple set of easily physically justified assumptions could explain the key features of the relationship NPH document: the sharp pickup associated with a peak in rainfall variance and a flattening of the humidity-rainfall relationship at higher humidity values. In this work, we introduce a simple, physically-based two-layer model that reproduces these features. Our model has some commonalities with NPH's interpretation, which we discuss further in our conclusions. However, we believe that the physical justification for the assumptions in our model is more straightforward. In addition, our model can explain the relationship between humidity and rainfall over a broader range of column water vapor than the power law fit that NPH base their interpretation on. Hence, our interpretation is less sensitive to very high humidity values at which satellite retrievals may be problematic.

2. Model Assumptions

[6] We firstly assume independent Gaussian distributions of humidity in the boundary layer and in the free troposphere. As motivation for this assumption, Figure 2 shows the probability density function of water vapor path averaged to 24 $\times$ 24 km resolution, below and above 850 mbar in a radiative convective equilibrium simulation with a Cloud Resolving Model (CRM). The CRM was run to statistical radiative convective equilibrium on a 1024 $\times$ 1024 km horizontal domain with 4 km horizontal resolution and 64 vertical levels, with specified radiative cooling consistent with a similar smaller domain 300 K surface temperature run. The wind was relaxed (time scale of two hours) towards a background wind profile with shear of 5 m/s linearly decreasing from the surface to 16 km. Details about the CRM are given by *Khairoutdinov and*

[1]Department of Earth, Atmospheric and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.

0094-8276/09/2009GL039667$05.00

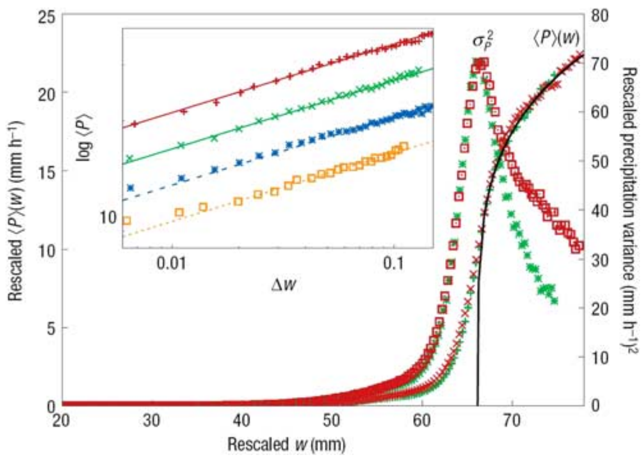

Figure 1. Figure 1 from *Peters and Neelin* [2006], showing precipitation rates and their variances versus column water vapor w for two regions of the tropical Pacific, as well as a power-law fit above the critical point (solid line). Both the precipitation and the water vapor path are rescaled by empirical constants so that the curves collapse. The inset shows on double-logarithmic scales the precipitation rate as a function of $(w - w_{critical})/w_{critical}$, where $w_{critical}$ is the critical water vapor path at which precipitation picks up. Reprinted by permission from Macmillan Publishers Ltd: [Nature Physics] [*Peters and Neelin*, 2006], copyright (2006).

Randall [2003]. We see that the Gaussian assumption is a reasonable approximation, although there are some departures from it. In the CRM, the boundary layer and free tropospheric water vapor paths are slightly correlated with a correlation coefficient of −0.2; in our model they are assumed to be independent, but this is not a crucial assumption (see section 4). Also the means of the boundary layer and free tropospheric water vapor paths differ, but this depends on the choice for the pressure cutoff between the two layers. For definiteness, we define each of our model layers as contributing roughly half of the column-integrated water vapor.

[7] Our second assumption is that precipitation occurs only when the lower layer water vapor exceeds a critical value (we use the term critical to be consistent with NPH's terminology, but here critical need not imply the presence of long-range correlations, scale-free behavior, etc. In our usage, the term simply refers to a threshold). In the tropics, horizontal temperature gradients are small due to the large Rossby deformation radius, so instability, as measured by convective available potential energy (CAPE), depends primarily on low-level humidity. Thus, our second model assumption corresponds loosely to assuming that rainfall does not occur below a critical CAPE value.

[8] The third assumption in our model is that when the lower-layer water vapor exceeds the critical value, precipitation is a linear function of column-integrated water vapor. We expect rainfall to be modulated by humidity at many levels [e.g., *Bretherton et al.*, 2004; *Holloway and Neelin*, 2009] for reasons we describe in more detail below. The linear functional form of the column humidity-rainfall relationship in rainy conditions is chosen for simplicity. This assumption is not crucial to the ideas underlying our model and could be modified beyond the scope of this paper.

[9] The assumption that when deep convection is occurring, more free tropospheric humidity leads to more rainfall can be rationalized in two ways. Most air parcels rising in deep convective updrafts are strongly diluted by mixing with environmental air, so their buoyancy is affected by moisture at many levels. When rising parcels entrain mois-

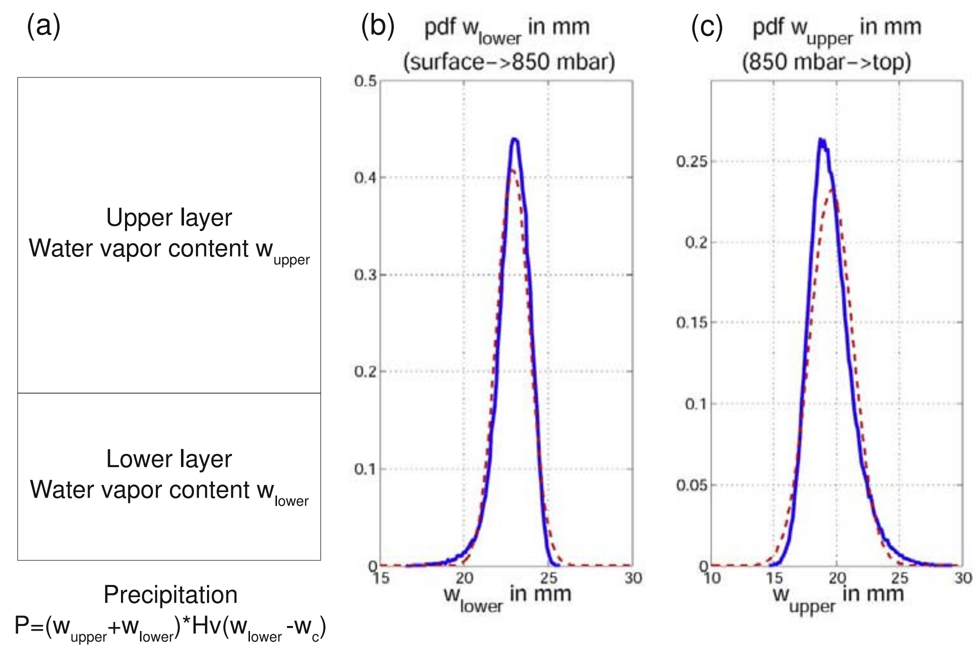

Figure 2. (a) Illustration of the two-layer model for precipitation. H_v denotes the Heaviside function. (b and c) Probability density function of water vapor (solid lines) below and above 850 mbar in a cloud-resolving model with a fixed sea surface temperature (SST) of 300 K. The dashed lines show Gaussian densities with the corresponding means and standard deviations.

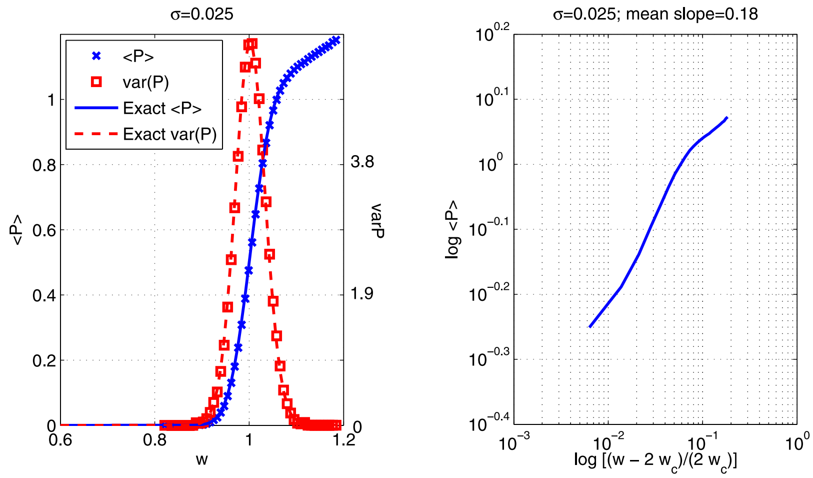

Figure 3. Precipitation and its variance versus column water vapor for $\mu = w_c = 0.5$ and $\sigma = 0.025$. The exact solution from equations (2) and (3) are also shown; they agree very well with the numerical solutions. (right) Precipitation versus $(w - 2w_c)/(2w_c)$. (left) For $\sigma = 0.025$, the mean log-log slope is ≈ 0.2.

ter environmental air, they remain positively buoyant longer, rise further and the convection is more vigorous than in drier conditions with similar temperature profiles. Alternatively, the moisture-precipitation relationship can also be justified using boundary layer quasi-equilibrium [*Raymond*, 1995; *Emanuel*, 1995], which postulates a balance between moistening of the boundary layer by surface evaporation and drying of the boundary layer by precipitation-driven cold pools. In moister conditions, fewer precipitation-driven downdrafts occur, so more deep convection and rainfall is needed to balance a given surface forcing.

[10] Our model is a promising alternative interpretation of NPH's observations, but at this stage we consider it a toy model. Further research, potentially using a cloud-resolving model (CRM) or observations, is needed to test and constrain the model in more detail before we would consider it a fully quantitative model of the relationship between rainfall and humidity.

3. Analytical Description and Exact Solution

[11] Our goal is to examine the relationship between a normalized precipitation P and column water vapor w in a system following the assumptions described above. We use a simple model of the atmosphere with two layers, whose water vapor paths are modeled by independent random variables w_{lower} and w_{upper}, as illustrated in Figure 2a. We assume that w_{lower} and w_{upper} are normally distributed with the same mean μ and standard deviation σ. In the following, w_{lower} and w_{upper} are normalized by the total column-integrated water vapor path so that the means of the upper layer and lower layer water vapor paths are both equal to $\mu = 0.5$.

[12] In our model, the precipitation is non-zero only if the low-layer water vapor exceeds a threshold value w_c. Then it is given by the total column water vapor

$$P = \left(w_{upper} + w_{lower}\right) H_v\left(w_{lower} - w_c\right), \tag{1}$$

where H_v denotes the Heaviside function, and where P is a non-dimensional precipitation, normalized by an arbitrary time scale.

[13] The expected value of precipitation for a given total column water vapor $w = w_{lower} + w_{upper}$ is therefore

$$\langle P\rangle(w) = \frac{w}{2}\,\mathrm{erfc}\left(\frac{w_c - w/2}{\sigma}\right), \text{ where } \mathrm{erfc}(X) \equiv \frac{2}{\sqrt{\pi}}\int_X^\infty e^{-t^2}\,dt, \tag{2}$$

and its variance

$$\mathrm{var}P(w) = \frac{w^2}{2}\,\mathrm{erfc}\left(\frac{w_c - w/2}{\sigma}\right)\left[1 - \frac{1}{2}\,\mathrm{erfc}\left(\frac{w_c - w/2}{\sigma}\right)\right]. \tag{3}$$

Interestingly, neither $\langle P\rangle$ nor the conditional variance depend on the mean μ. The mean only affects the probability density function of w, not the relationship between w and $\langle P\rangle$. Note that this would not be true if the means of the upper and lower water vapor paths were different.

[14] In the following section, we compare these exact solutions with results from Monte Carlo simulations. In the numerical simulations, we also enforce that both w_{upper} and w_{lower} are positive. But even with this slight difference, equations (2) and (3) are in very good agreement with the numerical solutions.

4. Results

[15] Figure 3 (left) shows the precipitation and its variance as a function of the column water vapor obtained from Monte Carlo simulations (the Monte Carlo simulations were performed with typical sample size 10^7, and the total column water vapor is forced to remain above 0) for $\mu = w_c = 0.5$ and for $\sigma = 0.025$. The value chosen for σ is based on a CRM simulation; from the probability density functions in Figure 2, an approximate value for the standard deviation is $\sigma \approx .05\mu$, hence $\mu = 0.5$ yields $\sigma \approx 0.025$. The exact solutions given in equations (2) and (3) are also shown, they agree very well with the numerical solutions. Despite its simplicity, our model predicts the pickup in precipitation around $w = 2w_c$, as well as the peak in variance for that value of σ (compare Figure 1 and Figure 3, left). The reason why the variance peaks at the critical water vapor path is straightforward: at low w it does not rain, and at high w it almost always rains, since it is hard to reach

high values of w without having $w_{lower} > w_c$. Therefore the largest variability in precipitation is expected between these two limits, i.e., near the critical value of w.

[16] We checked the sensitivity of our results to the various parameters of our model. The shape of precipitation versus water vapor relationship is suprisingly robust to parameter changes. As mentioned earlier, the results are independent of the mean μ (see equations (2) and (3)). In particular, the location of the pickup in precipitation, $w = 2w_c$, only depends on the low-layer critical water vapor, not on the mean μ. Changing w_c also does not affect the shape of $\langle P \rangle(w)$, it primarily shifts the location of the critical w, where precipitation picks up and its variance reaches a maximum. Similarly, the key features of the relationship $\langle P \rangle(w)$ do not depend on the standard deviation σ; varying σ makes the pickup in precipitation more or less localized near the critical water vapor path (the results for various values of σ are given in the auxiliary material).[1]

[17] One could also choose a different equation for the precipitation in equation (1), such that for instance the precipitation intensity only depends on the upper layer water vapor $\langle P \rangle(w) = w_{upper} H_v \, (w_{lower} - w_c)$. The shape of $\langle P \rangle(w)$ and of its variance found are almost identical in this case. Choosing equal means for w_{upper} and w_{lower} is also not crucial: allowing these means to be different only shifts the location of the critical w. If we relax the assumption that w_{upper} and w_{lower} are independent by allowing for a correlation between them, the results are still unchanged up to strong correlations above 0.8 or so in absolute value (the correlation in the CRM simulation is -0.2).

[18] The only change that does make a slight difference is if the low-layer water vapor is enforced to remain at or below its critical value, i.e. $w_{lower} = \min(w_{lower}, w_c)$. The main results are unchanged, but the pickup in precipitation is not as steep, and the variance does not decrease all the way to zero for w above the critical value.

[19] We note in passing that although there is no clear evidence for a power law dependence of precipitation at high w, the mean power is about 0.2 (see Figure 3, right), in agreement with NPH who derive a power 0.21–0.26 from data.

5. Conclusions

[20] We have shown that a very simple, physically motivated two-layer model can reproduce the observed relationship between column-integrated water vapor, precipitation and its variance, as shown by NPH. In our model, humidity in the boundary layer and in the free troposphere are assumed to be independent and rainfall occurs only when the boundary layer humidity exceeds a critical value, below which the atmosphere is assumed stable. The amount of rainfall then depends on column-integrated humidity.

[21] In this model, rainfall increases rapidly with column-integrated humidity close to a critical humidity, and the slope of this relationship decreases at very high humidities. Also, as in observations, rainfall variance as a function of humidity is maximum near the critical humidity. Qualitatively at least, our model explains the humidity-rainfall relationship over a wider range of humidities than NPH and associated work.

[22] Our model may also explain the result in NPH that the observed critical water vapor does not scale with column-integrated saturation humidity, but instead like the lower tropospheric saturation humidity (see Figure 3 of NPH). The location of the pickup in precipitation, $w = 2w_c$, only depends on the low-layer critical water vapor w_c. If we assume that w_c corresponds to a critical relative humidity r_c independent of temperature, then the pickup in precipitation occurs when the column-integrated water vapor is $w(T) = 2w_c(T) = 2\ r_c\ w_{lower,sat}(T)$, where $w_{lower,sat}(T)$ is the low-layer saturation water vapor and where T denotes temperature. Therefore, as temperature changes, the critical column-integrated water vapor in our model scales with the low-layer saturation humidity.

[23] *Peters and Neelin* [2006] used the observed relationship between humidity and rainfall (Figure 1) to argue that physics analogous to that occurring in continuous phase transitions are important to the dynamics of the moist atmosphere at the scales in question. Our model also contains an inherent singularity where lower tropospheric humidity approaches the critical value (dP/dw goes to infinity at this point). Our model is also not inconsistent with NPH's interpretation that the tropical atmosphere self-organizes toward a stability threshold like that postulated in our model. In this view, surface evaporation provides a slow, continuous forcing bringing the atmosphere towards this threshold, while convection events rapidly dissipate instability after it is generated (the quasi-equilibrium postulate [e.g., *Arakawa and Schubert*, 1974; *Emanuel et al.*, 1994]).

[24] However, in contrast to *Peters and Neelin* [2006], our model assumes that transition physics is unimportant and instead that the relevant transition can simply be described using a Heaviside function. As discussed in the introduction, we view this as physically corresponding to a stability threshold that a column must exceed in order for deep convection and rainfall to occur. Also, in contrast to NPH's interpretation, in our model, the high end of the water vapor-precipitation curve does not need to be universal. In fact, as seen in equation (2), this curve depends on the assumed probability density function of water vapor paths near the stability threshold. Our model formulation is also agnostic about time-space scaling (unlike the critical phenomenon analogy).

[25] In our view, the mere existence of a stability threshold and an approximate power-law behavior of rainfall near that threshold does not establish that tropical convection is an example of SOC. *Peters et al.* [2002] do present evidence that rainfall event sizes follow a power law distribution, which suggests that the theory of critical phenomena is relevant to atmospheric convection. Further evidence could come from detailed analysis of the spatio-temporal character of convection [e.g., *Cohen and Craig*, 2006; *Peters et al.*, 2009].

[26] **Acknowledgments.** Thanks to Eric Downes, David Neelin, Ole Peters, Daniel Rothman and Bo Yang for useful discussions on this work.

References

Arakawa, A., and W. H. Schubert (1974), Interaction of a cumulus cloud ensemble with the large–scale environment, part I, *J. Atmos. Sci.*, *31*, 674–701.

[1]Auxiliary materials are available in the HTML. doi:10.1029/2009GL039667.

Bony, S., and K. A. Emanuel (2005), On the role of moist processes in tropical intraseasonal variability: Cloud–radiation and moisture–convection feedbacks, *J. Atmos. Sci.*, *62*, 2770–2789.
Bretherton, C. S., M. E. Peters, and L. E. Back (2004), Relationships between water vapor path and precipitation over the tropical oceans, *J. Clim.*, *17*, 1517–1528.
Cohen, B. G., and G. C. Craig (2006), Fluctuations in an equilibrium convective ensemble. Part II: Numerical experiments, *J. Atmos. Sci.*, *63*, 2005–2015.
Emanuel, K. A. (1995), The behavior of a simple hurricane model using a convective scheme based on subcloud-layer entropy equilibrium, *J. Atmos. Sci.*, *52*, 3960–3968.
Emanuel, K. A., J. D. Neelin, and C. S. Bretherton (1994), On large–scale circulations in convecting atmospheres, *Q. J. R. Meteorol. Soc.*, *120*, 1111–1143.
Holloway, C. E., and J. D. Neelin (2009), Moisture vertical structure, column water vapor, and tropical deep convection, *J. Atmos. Sci.*, *66*, 1665–1683.
Khairoutdinov, M. F., and D. A. Randall (2003), Cloud resolving modeling of the ARM summer 1997 IOP: Model formulation, results, uncertainties, and sensitivities, *J. Atmos. Sci.*, *60*, 607–625.
Khouider, B., and A. J. Majda (2006), A simple multicloud parameterization for convectively coupled tropical waves. Part I: Linear analysis, *J. Atmos. Sci.*, *63*, 1308–1323.
Kuang, Z. (2008), A moisture-stratiform instability for convectively coupled waves, *J. Atmos. Sci.*, *65*, 834–854.
Neelin, J. D., and J.-Y. Yu (1994), Modes of tropical variability under convective adjustment and the Madden–Julian Oscillation. Part I: Analytical theory, *J. Atmos. Sci.*, *51*, 1876–1894.
Neelin, J. D., O. Peters, and K. Hales (2009), The transition to strong convection, *J. Atmos. Sci.*, in press.
Peters, O., and J. D. Neelin (2006), Critical phenomena in atmospheric precipitation, *Nat. Phys.*, *2*, 393–396.
Peters, O., C. Hertlein, and K. Christensen (2002), A complexity view of rainfall, *Phys. Rev. Lett.*, *88*, 018701, doi:10.1103/PhysRevLett.88.018701.
Peters, O., J. D. Neelin, and S. W. Nesbitt (2009), Mesoscale convective systems and critical clusters, *J. Atmos. Sci.*, in press.
Raymond, D. J. (1995), Regulation of moist convection over the west Pacific warm pool, *J. Atmos. Sci.*, *52*, 3945–3959.
Raymond, D. J. (2000), Thermodynamic control of tropical rainfall, *Q. J. R. Meteorol. Soc.*, *126*, 889–898.
Raymond, D. J., and Ž Fuchs (2009), Moisture modes and the Madden-Julian Oscillation, *J. Clim.*, *22*, 3031–3046.

L. E. Back, K. A. Emanuel, C. J. Muller, and P. A. O'Gorman, Department of Earth, Atmospheric and Planetary Sciences, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA. (mullerc@mit.edu)