

FUNDAMENTALS OF FLUID MECHANICS

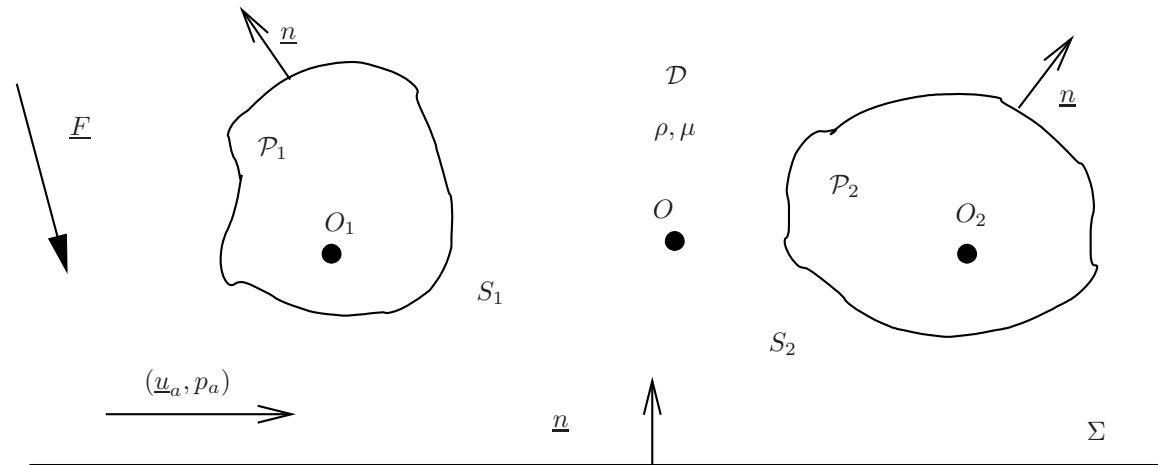
MMI103

Lesson 6 (course and exercises)

Thursday 15th October 2025

INCOMPRESSIBLE NAVIER-STOKES EQUATIONS FOR THE HOMOGENEOUS FLUID

$$\rho \left\{ \frac{\partial \underline{u}}{\partial t} + \underline{\text{grad}}[\underline{u}] \cdot \underline{u} \right\} = \rho \underline{F} - \underline{\text{grad}}[p] + \mu \Delta \underline{u}, \quad \text{div}(\underline{u}) = 0$$



- **Boundary conditions** to be prescribed
 - In each direction for which the fluid domain is not bounded and at the fluid domain boundaries (solid surface, droplet surface, bubble surface,...)
- **A tremendously-involved task: non linear equations, dissipative equation,...**

STOKES AND REYNOLDS NUMBERS

$$\rho \left\{ \frac{\partial \underline{u}}{\partial t} + \underline{\text{grad}}[\underline{u}] \cdot \underline{u} \right\} = \rho \underline{F} - \underline{\text{grad}}[p] + \mu \Delta \underline{u},$$

- Flow time scale τ , length scale L and velocity scale U
 - Typical sizes of the different occurring terms

$$\left| \rho \frac{\partial \underline{u}}{\partial t} \right| \sim \frac{\rho U}{\tau}, \quad \left| \rho \underline{\text{grad}}[\underline{u}] \cdot \underline{u} \right| \sim \frac{\rho U^2}{L}, \quad \left| \mu \Delta \underline{u} \right| \sim \frac{\mu U}{L^2}$$

- Stokes number St and Reynolds number Re

$$St = \frac{\rho L^2}{\mu \tau} = \frac{\tau_v}{\tau} \quad \text{with} \quad \tau_v = \frac{L^2}{\nu} \quad (\nu = \mu/\rho), \quad Re = \frac{\rho U L}{\mu}$$

- Comparing with the viscous (diffusion) term

$$\left| \frac{\rho \frac{\partial \underline{u}}{\partial t}}{\mu \Delta \underline{u}} \right| \sim St, \quad \left| \frac{\rho \underline{\text{grad}}[\underline{u}] \cdot \underline{u}}{\mu \Delta \underline{u}} \right| \sim Re$$

SOLVING THE NAVIER STOKES EQUATIONS?

- **Analytical (exact) solutions** in some simple cases (Couette and Poiseuille flows, other exercices)
- **Numerical solution.** Can be very involved and very cpu time consuming (especially for $\text{Re} \gg 1$)
- Incompressible ideal (non-viscous) fluid. **Euler equations**

$$\text{Re} = \frac{\rho U L}{\mu} \gg 1, \quad \rho \frac{\partial \underline{u}}{\partial t} + \underline{\underline{\text{grad}}}[\underline{u}] \cdot \underline{u} = \rho \underline{F} - \underline{\text{grad}}[p]$$

- Incompressible creeping flow. **Stokes equations**

$$\text{Re} = \frac{\rho U L}{\mu} \ll 1, \quad \rho \frac{\partial \underline{u}}{\partial t} = \rho \underline{F} - \underline{\text{grad}}[p] + \mu \Delta \underline{u}$$

VORTICITY

- Vorticity: $\underline{w} = \underline{\text{rot}}(\underline{u})$
- Governing equation for the vorticity?

$$\rho \left\{ \frac{\partial \underline{u}}{\partial t} + \underline{\text{rot}}(\underline{u}) \wedge \underline{u} + \underline{\text{grad}}[u^2/2] \right\} = \rho \underline{F} - \underline{\text{grad}}[p] + \mu \Delta \underline{u}, \quad \text{div}(\underline{u}) = 0$$

- Mathematical identity

$$\underline{\text{rot}}[\underline{\text{rot}}(\underline{a})] = \underline{\text{grad}}[\text{div}(\underline{a})] - \Delta \underline{a}$$

- Since $\text{div}(\underline{u}) = \text{div}(\underline{w}) = 0$ then $\underline{\text{rot}}(\Delta \underline{u}) = \Delta \underline{w}$
 - Resulting local vorticity conservation law

$$\frac{\partial \underline{w}}{\partial t} + \underline{\text{rot}}(\underline{w} \wedge \underline{u}) = \nu \Delta \underline{w} + \underline{\text{rot}}(\underline{F}), \quad \nu = \mu/\rho$$

- **Consequence:** for the flow to be **potential** (i.e. $\underline{u} = \underline{\text{grad}}[\phi]$ so that $\underline{w} = \underline{0}$) the body forces \underline{F} must satisfy $\underline{\text{rot}}(\underline{F}) = \underline{0}$

POTENTIAL FLOW

- The flow is **potential** when there exists a velocity potential ϕ with

$$\underline{u} = \underline{\text{grad}}[\phi], \quad (\underline{w} = \underline{0})$$

- **First consequence 1:** the body force is **conservative**

$$\underline{F} = -\underline{\text{grad}}[\psi_m] \quad (\underline{\text{rot}}(\underline{F}) = \underline{0}), \quad \psi_m = \psi_m(\underline{x}, t)$$

- **Consequence 2:** for a non-viscous fluid the **second Bernoulli theorem**

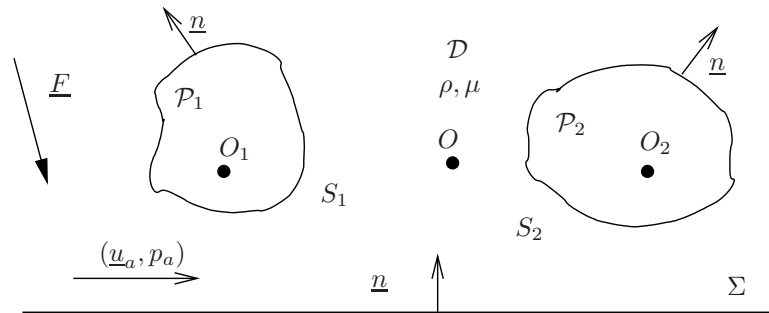
$$p = C(t) - \rho \left\{ \frac{\partial \phi}{\partial t} + \underline{\text{grad}}[\phi] \cdot \underline{\text{grad}}[\phi] / 2 + \psi_m(\underline{x}, t) \right\}$$

- **Remarks:**

- (i) \underline{u} : **linearly** depends upon ϕ
- (ii) p : **non-linear** dependence upon ϕ
- (iii) for given ψ_m one only needs to determine ϕ

GOVERNING PROBLEM FOR A VELOCITY POTENTIAL

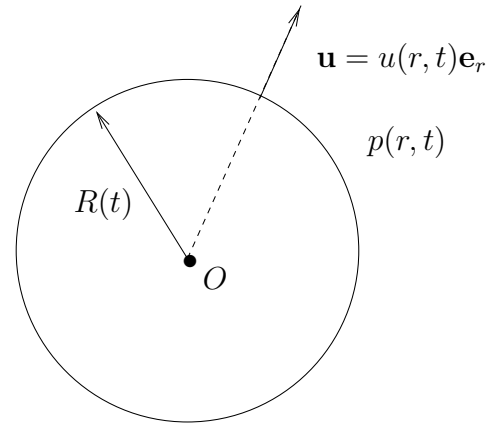
$$\Delta\phi = 0$$



- **Boundary conditions** which can be non-linear in ϕ
 - For directions in which the fluid is unbounded: $\underline{u} = \underline{\text{grad}}[\phi]$ and p given
- At the **impermeable** boundaries (solid, bubble, droplets,...) with normal \underline{n} pointing into the fluid and velocity \underline{w}

$$\underline{\text{grad}}[\phi] \cdot \underline{n} = \underline{w} \cdot \underline{n}$$

EXAMPLES OF POTENTIAL FLOWS WITH SPHERICAL SYMMETRY



- Assumptions: $p_\infty(t)$ is prescribed and

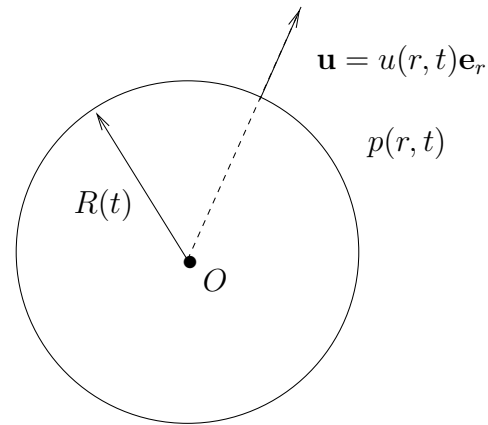
$$\underline{U} = u(r, t)\underline{e}_r, \quad p = p(r, t), \quad \underline{F} = \underline{0}$$

- Then the flow is **potential** and the solution reads

$$\phi = -A(t)/r, \quad u = A(t)/r^2, \quad C(t) = p_\infty(t), \quad A(t) = R^2 u(R, t)$$

Moreover, for a **non-viscous** fluid the **second Bernoulli theorem** gives

$$p(r, t) = p_\infty(t) - \rho \left\{ \frac{A^2(t)}{2r^4} - \frac{A'(t)}{r} \right\}$$



- Associated stress tensor

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \mu \left\{ 2 \frac{\partial u}{\partial r} \underline{\underline{e}}_r \otimes \underline{\underline{e}}_r + \frac{u}{r} (\underline{\underline{I}} - \underline{\underline{e}}_r \otimes \underline{\underline{e}}_r) \right\}$$

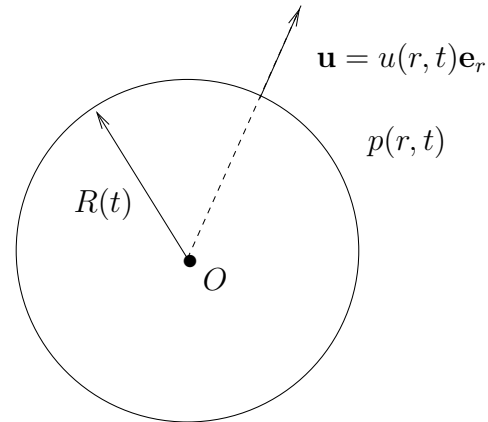
- Resulting local stress on the sphere surface

$$\underline{\underline{T}} = \underline{\underline{\sigma}} \cdot \underline{\underline{e}}_r = - \left[p(R, t) + 4\mu \frac{A(t)}{R^3} \right] \underline{\underline{e}}_r$$

- Consequently

- (i) the flow exerts on the sphere a **radial** stress
- (ii) zero net force and torque (about the sphere center O)

FIRST EXAMPLE: SOLID POROUS SPHERE WITH FLUID INJECTION OR ASPIRATION



- R is **time-independent** and provided. Again $p_\infty(t)$ is provided
- The injection-aspiration velocity $W(t) = u(R, t) = A(t)/R^2$ is prescribed
- **The velocity** therefore is (since $A(t) = R^2W(t)$ is known)

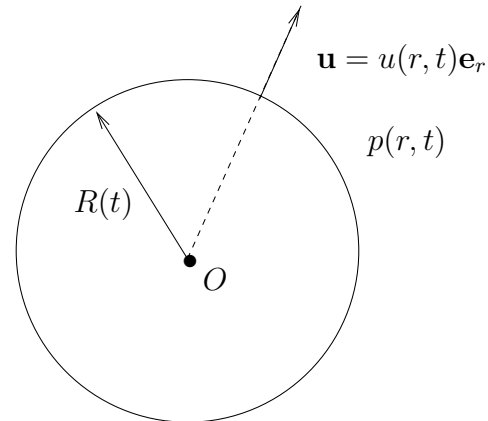
$$u(r, t) = R^2W(t)/r^2$$

- The pressure follows. For a non-viscous fluid (see later for viscous one)

$$p(r, t) = p_\infty(t) - \rho\left\{\frac{R^4W^2(t)}{2r^4} - \frac{R^2W'(t)}{r}\right\}$$

- **Exact** solution of the **unsteady and non-linear** incompressible Navier-Stokes equations!

SECOND EXAMPLE: OSCILLATING SPHERICAL BUBBLE WITH UNIFORM SURFACE TENSION γ



- The spherical bubble radius $R(t)$ is unknown. Here, $u[R, t] = R'(t)$
 - Uniform pressure p_b inside the bubble with

$$p_b = p_v + G/R^3(t)$$

- At the spherical bubble the 'Laplace' law gives

$$\underline{T} \cdot \underline{e}_r = [2\mu \frac{\partial u}{\partial r} - p](R, t) = \frac{2\gamma}{R} - p_b$$

The fluid is viscous. One gets the pressure and then the Rayleigh-Plesset equation

- Indeed, taking u after some algebra one gets

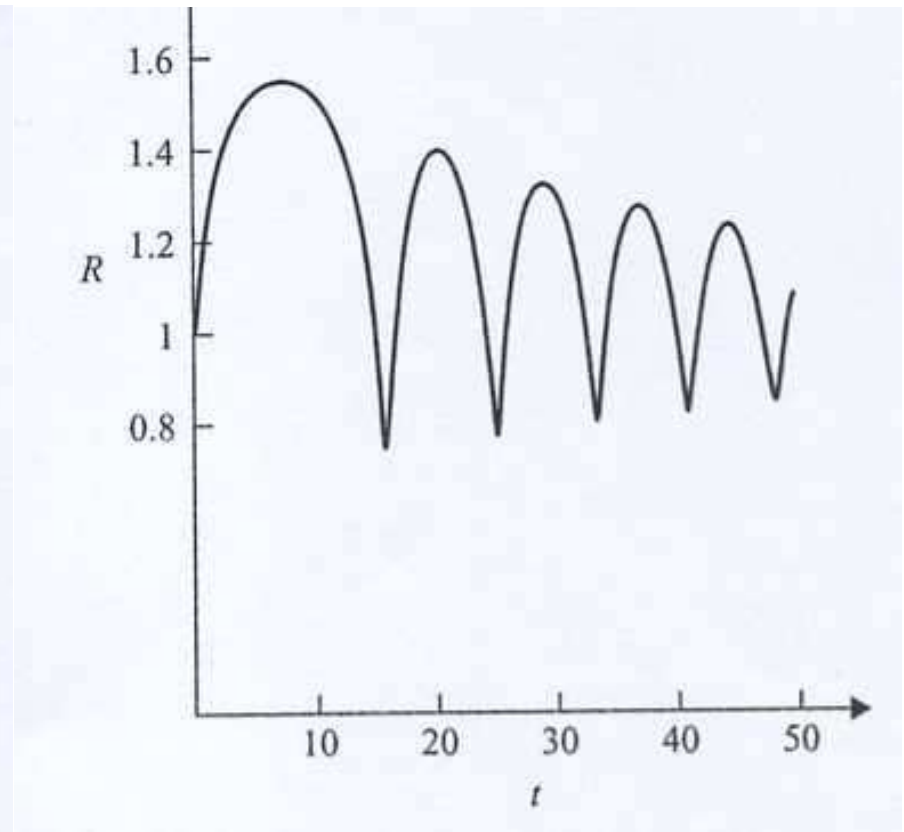
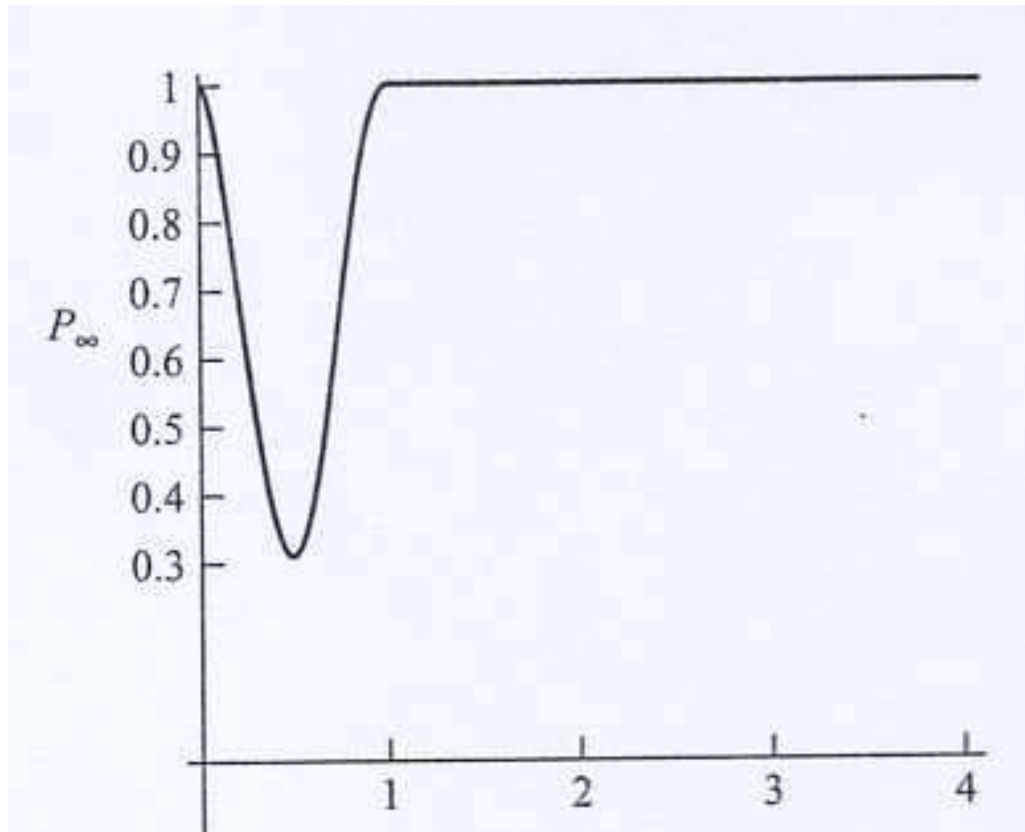
$$RR'' + \frac{3}{2}(R')^2 = \frac{p_v - p_\infty(t)}{\rho} + \frac{1}{\rho} \left(\frac{G}{R^3} - \frac{2\gamma}{R} - \frac{4\mu R'}{R} \right)$$
$$R(t=0) = R_e$$

- The far-field pressure $p_\infty(t)$ is prescribed
 - Need to also provide $R'(t=0)$
 - **Non-linear** equation!
- The case of a bubble at rest is retrieved
($R' = 0$ et $p_\infty(t) = p_l$ constante)

Quite a very few analytical solutions!

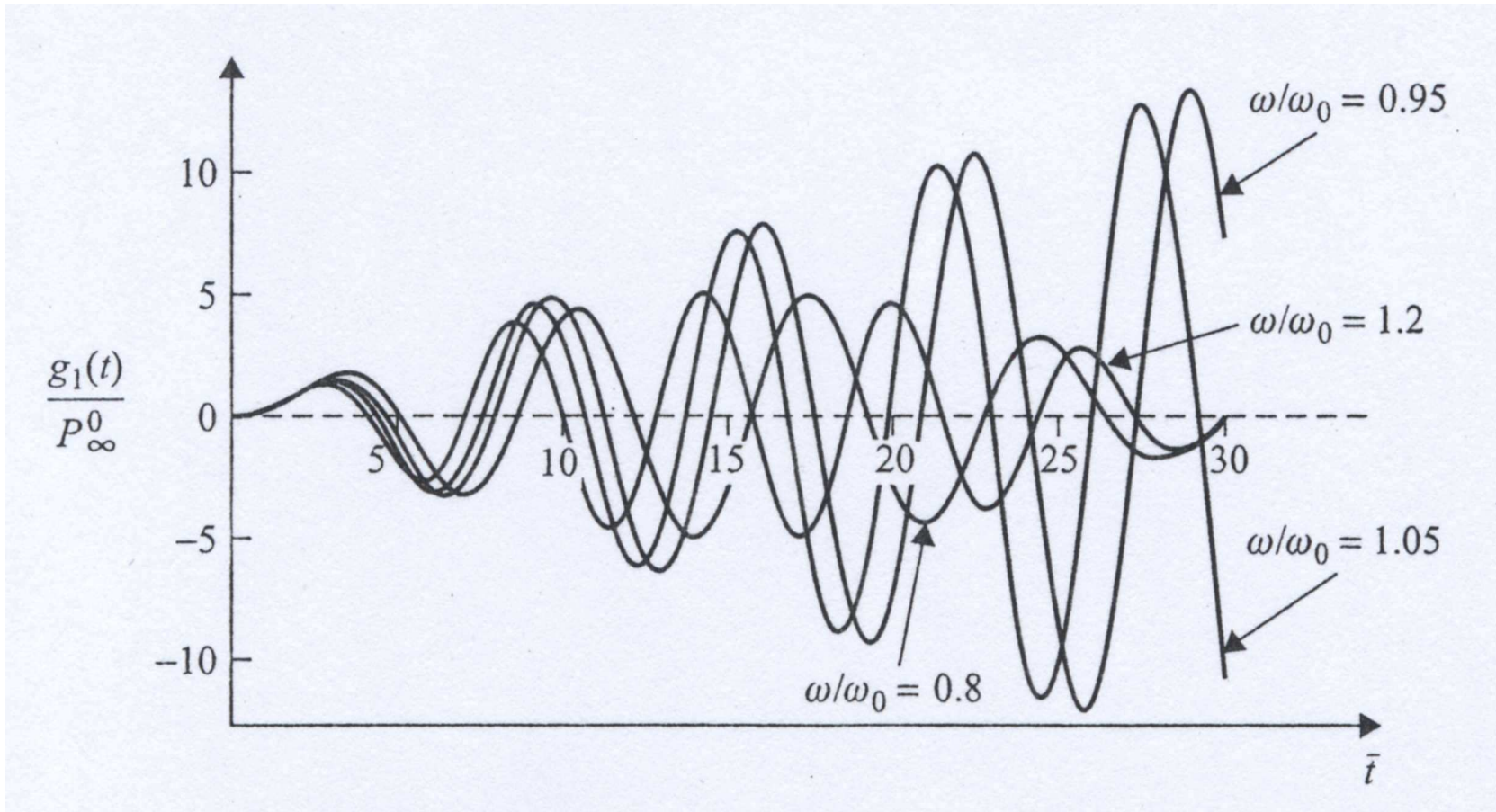
Quite different bubble behaviours depending on the applied far-field pressure $p_\infty(t)$

First example



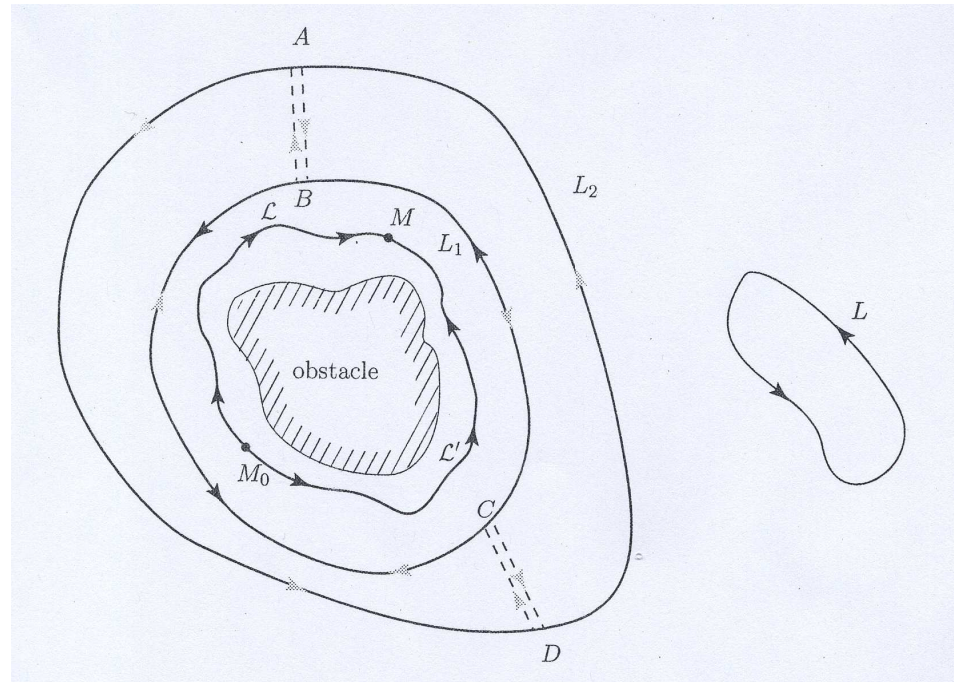
- Large relative changes of the far-field pressure
 - The bubble oscillates!
- Quite different time scales!!

Second Example



- Modulated far-field pressure $p_\infty(t) = P_\infty^0[1 + \epsilon \sin \omega t]$
 - Oscillating bubbles! With $R(t) = R_e[1 + \epsilon g_1(t)]$
 - Resonance!

Circulation along a closed contour



$$\Gamma_L = \oint_L \underline{U} \cdot \underline{dl}$$

For each surface Σ sustained by the closed contour L with the right orientation for the unit normal \underline{n} on Σ

$$\Gamma_L = \int_{\Sigma} \underline{\text{rot}}(\underline{U}) \cdot \underline{n} dS = \int_{\Sigma} \underline{w} \cdot \underline{n} dS$$

Circulation conservation law

- **Material tracking** of the circulation on a line (closed or not)

$$\frac{d}{dt} \int_{C(t)} \underline{a} \cdot \underline{dl} = \int_{C(t)} \left\{ \frac{d\underline{a}}{dt} \cdot \underline{dl} + \underline{a} \cdot (\text{grad}[\underline{U}] \cdot \underline{dl}) \right\}$$

- Application to the circulation on a **closed** line

$$\frac{d}{dt} \oint_{L(t)} \underline{U} \cdot \underline{dl} = \oint_{L(t)} \frac{d\underline{U}}{dt} \cdot \underline{dl} + \oint_{L(t)} \underline{U} \cdot (\text{grad}[\underline{U}] \cdot \underline{dl})$$

- Fundamental local momentum conservation law

$$\frac{d\underline{U}}{dt} = \underline{F} - \underline{\text{grad}}[p]/\rho + \nu \Delta \underline{U}, \quad \nu = \mu/\rho$$

- **Basic final result**

$$\frac{d}{dt} \oint_{L(t)} \underline{U} \cdot \underline{dl} = \oint_{L(t)} \underline{F} \cdot \underline{dl} + \nu \oint_{L(t)} \Delta \underline{U} \cdot \underline{dl}$$

KELVIN THEOREM

- previous result

$$\frac{d}{dt} \oint_{L(t)} \underline{U} \cdot \underline{dl} = \oint_{L(t)} \underline{F} \cdot \underline{dl} + \nu \oint_{L(t)} \underline{\Delta U} \cdot \underline{dl}$$

- Reminder: **conservative** body forces if

$$\underline{F} = -\underline{\text{grad}}[\psi_m], \quad \underline{\text{rot}}(\underline{F}) = \underline{0}$$

- Kelvin theorem

The circulation on any **material and closed** line is **conserved** when tracking the line in the flow for any flow of a viscous and homogeneous fluid subject to **conservatives** body forces

LAGRANGE THEOREM

- relation

$$\frac{d}{dt} \oint_{L(t)} \underline{U} \cdot \underline{dl} = \oint_{L(t)} \underline{F} \cdot \underline{dl} + \nu \oint_{L(t)} \underline{\Delta U} \cdot \underline{dl}$$

- Remind that

$$\oint_{L(t)} \underline{U} \cdot \underline{dl} = \int_{\Sigma} \underline{\text{rot}}(\underline{U}) \cdot \underline{n} dS = \int_{\Sigma} \underline{w} \cdot \underline{n} dS$$

- Lagrange theorem

A flow of a non-viscous and homogeneous fluid subject to conservative body forces and irrotational at a given time then remains irrotational as time evolves