

FUNDAMENTALS OF FLUID MECHANICS

MMI103

(course and exercises)

Thursdays 2th and 9th October 2025

NEWTONIAN FLUID

- Newtonian fluid: definition
- Newtonian fluid: case of the incompressible flow

INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

- Incompressible Navier-Stokes equations
- Case of a Newtonian **homogeneous** fluid

ANALYTICAL SOLUTIONS FOR UNIDIRECTIONAL FLOWS

- Unidirectional flow
 - Couette flow
 - Poiseuille flow

NEWTONIAN FLUID

- Fluid: $\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}$ with $\underline{\underline{\tau}}$ the 'viscous' part of the stress tensor

- **Newtonian fluid?** $\underline{\underline{\tau}} = \lambda_v \operatorname{div}(\underline{\underline{u}})\underline{\underline{I}} + 2\mu\underline{\underline{d}}$

$\underline{\underline{I}}$ is the usual identity tensor

$\underline{\underline{d}}$ is the so-called rate-of-strain tensor such that

$$\underline{\underline{d}} = (\operatorname{grad}[\mathbf{u}] + {}^t \operatorname{grad}[\mathbf{u}])/2$$

$\lambda_v \geq 0$ is the **volume viscosity**, $\mu \geq 0$ is the **shear viscosity**
(both are positive, due to the second law of thermodynamics)

- In Cartesian coordinates (x_1, x_2, x_3)

$$\underline{\underline{u}} = u_i \underline{\underline{e}}_i, \quad u_{i,j} = \frac{\partial u_i}{\partial x_j}, \quad \operatorname{div}(\underline{\underline{u}}) = u_{k,k}$$

$$I_{ij} = \delta_{ij}, \quad d_{ij} = (u_{i,j} + u_{j,i})/2$$

$$\sigma_{ij} = [-p + \lambda_v u_{k,k}] \delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

INCOMPRESSIBLE FLOW OF A NEWTONIAN FLUID

- In practice a **liquid**. Not necessarily homogenous! $\mu = \mu(\rho, T)$
 - Then only the shear viscosity μ is involved since

$$\text{div}(\underline{u}) = 0, \quad \underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu\underline{\underline{d}}$$

- Meaning of the shear viscosity $\mu \geq 0$? Consider the fluid flow

$$\underline{u} = \gamma x_3 \underline{e}_1$$

which is a **shear flow with rate of shear** γ

- For this shear flow the surface stress is, for $\underline{n} = \underline{e}_3$,

$$\underline{f} = \underline{\underline{\sigma}} \cdot \underline{n} = \mu\gamma \underline{e}_1 - p\underline{e}_3$$

Hence there is a **tangential stress** (here $\mu\gamma \underline{e}_1$) termed the **shear stress**

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INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

- μ is assumed to be **constant** (usually valid for liquids)

$$\operatorname{div}(\underline{u}) = 0, \quad \rho \frac{d\underline{u}}{dt} = \rho \underline{F} - \underline{\operatorname{grad}}[p] + \mu \Delta \underline{u}$$

- Proof? Use of Cartesian coordinates and of the identities

$$\operatorname{div}(\underline{u}) = u_{k,k}, \quad \operatorname{div}(\underline{\underline{\sigma}}) = \sigma_{ij,j} \underline{e}_i, \quad \sigma_{ij} = -p \delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

- Incompressible Navier-Stokes equations in Cartesian coordinates

$$u_{i,i} = 0, \quad \rho \left(\frac{\partial u_i}{\partial t} + u_j u_{i,j} \right) = \rho F_i - p_{,i} + \mu \Delta u_i, \quad \Delta g = g_{,kk}, \quad \Delta \underline{u} = (\Delta u_i) \underline{e}_i$$

- **Non-linear equations**

- Unknown Eulerian fields: (ρ, p, \underline{u}) . If the fluid is non-homogeneous need to add the energy conservation law (i.e. the first law of thermodynamics)

HOMOGENEOUS NEWTONIAN FLUID

- ρ constant and (p, \underline{u}) given from

$$\text{div}(\underline{u}) = 0, \quad \rho \frac{d\underline{u}}{dt} = \rho \underline{F} - \underline{\text{grad}}[p] + \mu \Delta \underline{u}$$

SOME USUAL BOUNDARY CONDITIONS

- On a solid impermeable **no-slip** boundary Σ moving at velocity \underline{w}

$$\underline{u} = \underline{w} \text{ on } \Sigma$$

- On a solid impermeable **slip** boundary Σ moving at velocity \underline{w} and having **slip length** $\lambda > 0$

$$\underline{u} = \underline{w} + \frac{\lambda}{\mu} [\underline{\underline{\sigma}} \cdot \underline{n} - (\underline{n} \cdot \underline{\underline{\sigma}} \cdot \underline{n}) \underline{n}] \text{ on } \Sigma$$

- Interface Σ between two **immiscible** fluids?

If plane or/and in absence of surface tension

$$\underline{u}_1 = \underline{u}_2 \text{ or } \underline{u}_1 \cdot \underline{N} = \underline{u}_2 \cdot \underline{N} \text{ on } \Sigma, \quad \underline{\underline{\sigma}}_1 \cdot \underline{n} = \underline{\underline{\sigma}}_2 \cdot \underline{n} \text{ on } \Sigma$$

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UNIDIRECTIONAL FLOW OF AN HOMOGENEOUS NEWTONIAN FLUID

- Unidirectional flow

$$\underline{u} = u(y, z, t)\underline{e}_x, \quad \underline{F} = \underline{0}$$

- resulting properties

$$\text{div}(\underline{u}) = 0, \quad \frac{d\underline{u}}{dt} = \frac{\partial \underline{u}}{\partial t}$$

- Navier-Stokes momentum conservation law reads

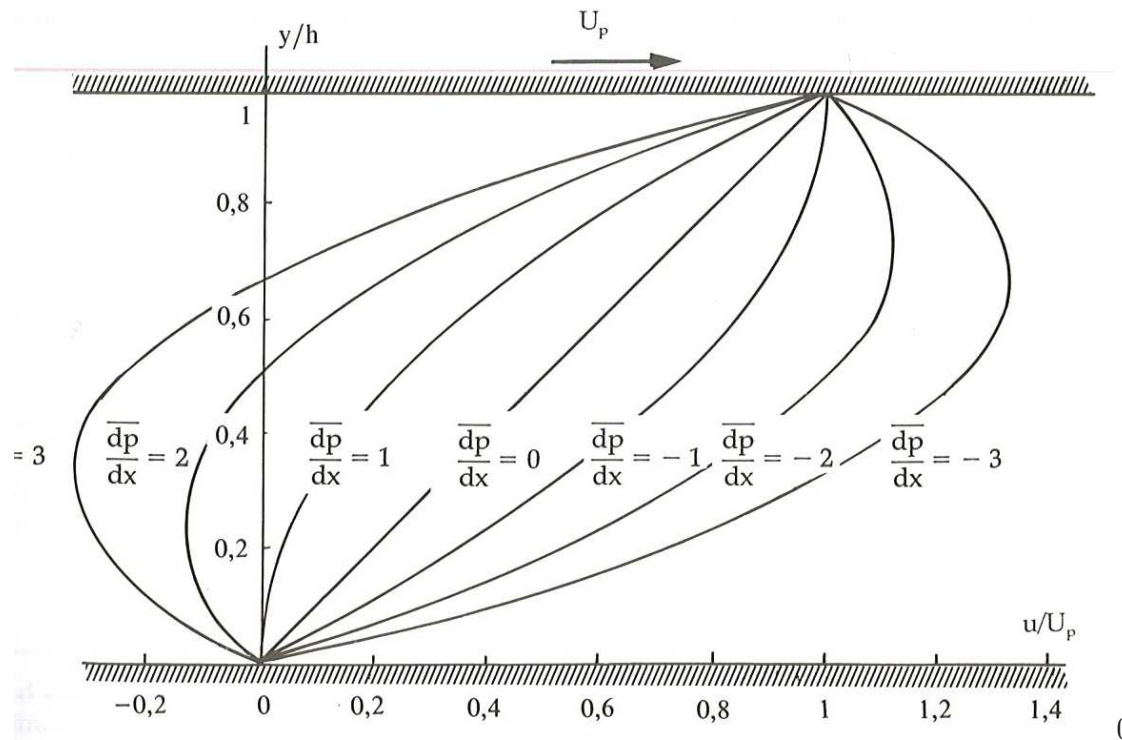
$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0, \quad \mu\Delta u - \rho\frac{\partial u}{\partial t} = \frac{\partial p}{\partial x}$$

- Solution of the following form

$$p = p(x, t), \quad \frac{\partial p}{\partial x} = C(t) = \mu\Delta u - \rho\frac{\partial u}{\partial t}$$

- Determination enforcing the proper boundary conditions for \underline{u} and p

STEADY UNIDIRECTIONAL FLOW BETWEEN TWO PARALLEL SOLID PLANES AT REST



Fluid domain: $0 \leq y \leq h$ with given $h > 0$

$$\underline{u} = u(y)\underline{e}_x, \quad u(0) = U_p, \quad u(h) = U_p$$

STEADY UNIDIRECTIONAL FLOW BETWEEN TWO PARALLEL SOLID **NO-SLIP** PLANES AT REST

- The previous problem formulation becomes

$$p = p(x), \quad \frac{dp}{dx} = C, \quad \frac{d^2u}{dy^2} = C/\mu$$

- The analytical solution is of the following form

$$u(y) = \frac{C}{2\mu}[y^2 + Ay + B]$$

- Here A and B are easily obtained from the **no-slip** velocity boundary conditions at $y = 0$ and $y = h$

$$A = \frac{1}{h}\left[\frac{2\mu(U_p - U_{p0})}{C} - h^2\right], \quad B = \frac{2\mu}{C}U_{p0}$$

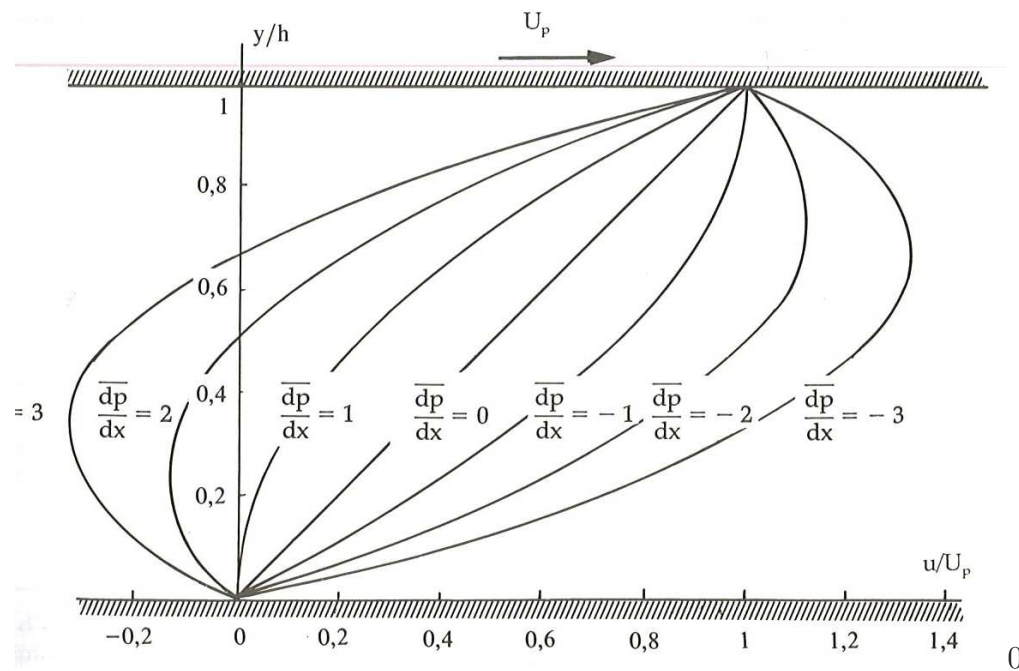
- The obtained final analytical solution reads

$$u(y) = U_{p0}\left(1 - \frac{y}{h}\right) + \frac{y}{h}U_p - \frac{Ch^2}{2\mu}\left(\frac{y}{h}\right)\left[1 - \frac{y}{h}\right], \quad p = Cx + cste$$

COUETTE FLOW

- Obtained for $U_{p0} = 0$ and $U_p \neq 0$

$$u(y) = U_p \left[\frac{y}{h} - \frac{d\bar{p}}{dx} \left(\frac{y}{h} \right) \left(1 - \frac{y}{h} \right) \right], \quad \frac{d\bar{p}}{dx} = \frac{h^2}{2\mu U_p} \frac{dp}{dx}$$



POISEUILLE FLOW

- Obtained for $U_{p0} = U_p = 0$

$$u(y) = -\frac{h^2}{2\mu} \frac{dp}{dx} \left(\frac{y}{h}\right) \left(1 - \frac{y}{h}\right)$$

